ANALYSIS AND ALGORITHMS OF BIFUZZY SYSTEMS

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A fuzzy variable is a function from a possibility space to the set of real numbers, while a bifuzzy variable is a function from a possibility space to the set of fuzzy variables. In this paper, a concept of chance distribution is originally presented for bifuzzy variable, and the linearity of expected value operator of bifuzzy variable is proved. Furthermore, bifuzzy simulations are designed and illustrated by some numerical experiments.

Keywords: Fuzzy variable; Bifuzzy variable; Chance distribution; Bifuzzy simulation.

1. Introduction

Since its introduction in 1965 by Zadeh, 1 fuzzy set theory has been well developed and applied in a wide variety of real problems. The term fuzzy variable was first introduced by Kaufmann, 2 and then defined by Nahmias as a function from a possibility space to the real line. 3

In many cases, the information about fuzzy sets cannot be given precisely. As a result, some researchers turned their attention to the extension of fuzzy sets. On the one hand, a few people discussed fuzzy sets when information on degree of membership is incomplete. For example, Zadeh introduced the concept of type 2 fuzzy set, 4 which is characterized by a fuzzy membership function. That is, the membership value for each element of a type 2 fuzzy set is a fuzzy set in [0,1]. Type 2 fuzzy set has also been studied widely. 5678 In 1983, twofold fuzzy set was derived by Dubois and Prade from possibility and necessity measures as a pair of fuzzy sets $A^+$ and $A^-$, 9 where $A^+$ represents the set of objects which possibly satisfy a non-vague property, and $A^-$ represents the set of objects which necessarily satisfy the non-vague property. Intuitionistic fuzzy set was initiated by Atanassov as a pair of membership functions $\mu^+$ and $\mu^-$ such that $\mu^+(x) + \mu^-(x) \leq 1$ for all $x$, 10 and then
studied from entropy, distances, operations.\textsuperscript{1112131415} Note that the intuitionistic fuzzy set was also called bifuzzy set. Furthermore, the set of membership value was extended to that of complex numbers by Nguyen \textit{et al.} so as to formalize some paradoxes.\textsuperscript{16}

On the other hand, some researchers considered fuzzy set when information on objects of the universal is incomplete. For example, a new concept, \textit{bifuzzy variable}, was defined by Liu as a function from a possibility space to the set of fuzzy variables. \textsuperscript{17} Some important concepts including the expected value operator and the chance measure were also defined. \textsuperscript{17}

As a continuation of these works, this paper investigates the mathematical properties of bifuzzy variable. The paper is organized as follows. After recalling some definitions and results about possibility space and credibility measure in Section 2, the concept of bifuzzy variable is introduced and extended to bifuzzy vector in Section 3. Then Section 4 discusses properties of the chance measure, and some new types of chance measure are suggested. Based on the concept of the chance measure, the chance distribution is originally presented for bifuzzy variable in Section 5. In order to measure bifuzzy variables, Section 6 introduces the expected value operator and proves its linearity. Following that, some other definitions are developed in Section 7 as a complementarity to measure bifuzzy variables including variance, covariance and moments. Furthermore, two theorems on the optimistic and pessimistic values are given in Section 8. Finally, in order to calculate expected value, find critical values, and compute chance functions in bifuzzy systems, some bifuzzy simulations are designed and illustrated by numerical experiments in Section 9.

2. Preliminaries

In order to provide an axiomatic theory to describe fuzziness, Nahmias suggested a theoretical framework. \textsuperscript{3} Let $\Theta$ be a nonempty set, and $P(\Theta)$ be the power set of $\Theta$. For each set $A \in P(\Theta)$, there is a nonnegative number $\text{Pos}\{A\}$, called its possibility, such that

(i) $\text{Pos}\{\emptyset\} = 0$, $\text{Pos}\{\Theta\} = 1$; and
(ii) $\text{Pos}\{\bigcup_k A_k\} = \sup_k \text{Pos}\{A_k\}$ for any arbitrary collection $\{A_k\}$ in $P(\Theta)$.

The triplet $(\Theta, P(\Theta), \text{Pos})$ is called a \textit{possibility space}, and the function $\text{Pos}$ is referred to as a possibility measure. The concept of \textit{credibility measure} $\text{Cr}$ was initiated by Liu and Liu as the average of possibility and necessity, \textsuperscript{18} i.e., $\text{Cr}\{A\} = (\text{Pos}\{A\} + \text{Nec}\{A\})/2$ for any set $A \in P(\Theta)$, where $\text{Nec}\{A\} = 1 - \text{Pos}\{A^c\}$. Note that the credibility measure is an increasing set function and has self duality and subadditivity, i.e.,

\[ \text{Cr}\{A\} + \text{Cr}\{A^c\} = 1, \quad \text{Cr}\{A \cup B\} \leq \text{Cr}\{A\} + \text{Cr}\{B\} \quad (1) \]

for any sets $A, B \in P(\Theta)$. 
Lemma 1. (Liu,\textsuperscript{19} Credibility Semicontinuity Law) Let $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ be a possibility space, and $A_1, A_2, \cdots, A \in \mathcal{P}(\Theta)$. We have $\lim_{i \to \infty} \text{Cr}(A_i) = \text{Cr}(A)$ if one of the following conditions is satisfied: (a) $\text{Cr}(A) \leq 0.5$ and $A_i \uparrow A$; (b) $\lim_{i \to \infty} \text{Cr}(A_i) < 0.5$ and $A_i \uparrow A$; (c) $\text{Cr}(A) \geq 0.5$ and $A_i \downarrow A$; (d) $\lim_{i \to \infty} \text{Cr}(A_i) > 0.5$ and $A_i \downarrow A$.

Suppose that $\xi$ is a fuzzy variable defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$. Based on the concept of credibility measure, Liu defined the credibility distribution $\Phi : [-\infty, +\infty] \to [0, 1]$ of $\xi$ as $\Phi(x) = \text{Cr}\{\theta \in \Theta|\xi(\theta) \leq x\}$.\textsuperscript{20}

Lemma 2. (Liu\textsuperscript{19}) The credibility distribution $\Phi$ of a fuzzy variable is a nondecreasing function on $[-\infty, +\infty]$ with

\[
\begin{align*}
\Phi(-\infty) &= 0 \\
\Phi(+\infty) &= 1 \\
\lim_{x \to -\infty} \Phi(x) &\leq 0.5 \leq \lim_{x \to +\infty} \Phi(x) \\
\lim_{y \downarrow x} \Phi(y) &= \Phi(x) \text{ if } \lim_{y \downarrow x} \Phi(y) > 0.5 \text{ or } \Phi(x) \geq 0.5.
\end{align*}
\]

Conversely, if $\Phi : [-\infty, +\infty] \to [0, 1]$ is a nondecreasing function satisfying (2), then $\Phi$ is the credibility distribution of fuzzy variable defined by the membership function

\[
\mu(x) = \begin{cases} 
2\Phi(x), & \text{if } \Phi(x) < 0.5 \\
1, & \text{if } \lim_{y \downarrow x} \Phi(y) < 0.5 \leq \Phi(x) \\
2 - 2\Phi(x), & \text{if } 0.5 \leq \lim_{y \downarrow x} \Phi(y).
\end{cases}
\]

In addition, Liu and Liu\textsuperscript{18} provided a general definition of expected value operator of fuzzy variable by

\[
E[\xi] = \int_{0}^{+\infty} \text{Cr}(\xi \geq r)dr - \int_{-\infty}^{0} \text{Cr}(\xi \leq r)dr
\]

provided that at least one of the above two integrals is finite.

3. Bifuzzy Variable

Definition 1. (Liu\textsuperscript{17}) A bifuzzy variable is a function from a possibility space to the set of fuzzy variables.
Example 1. Let \( \eta_1, \eta_2, \cdots, \eta_m \) be fuzzy variables and \( u_1, u_2, \cdots, u_m \) be real numbers in \([0,1]\) such that \( u_1 \lor u_2 \lor \cdots \lor u_m = 1 \). Then

\[
\xi = \begin{cases} 
\eta_1 \text{ with possibility } u_1 \\
\eta_2 \text{ with possibility } u_2 \\
\cdots \\
\eta_m \text{ with possibility } u_m
\end{cases}
\]

is clearly a bifuzzy variable.

Example 2. Let \( \xi = (\rho - 1, \rho, \rho + 1, \rho + 2) \), where \( \rho \) is a fuzzy variable with membership function \( \mu_\rho(x) = [1 - |x - 2|] \lor 0 \). Then \( \xi \) is a bifuzzy variable.

Example 3. The prediction of grain yield could be a bifuzzy variable, for example,

\[
\xi = \begin{cases} 
\text{"about 10000 ton" with possibility 0.6} \\
\text{"about 10500 ton" with possibility 0.8} \\
\text{"about 11200 ton" with possibility 1.0} \\
\text{"about 12000 ton" with possibility 0.7.}
\end{cases}
\]

Example 4. It is assumed that most people are middle. Then the height of a person can be described by

\[
\xi = \begin{cases} 
\text{"middle" with possibility 1.0} \\
\text{"tall" with possibility 0.8} \\
\text{"short" with possibility 0.6}
\end{cases}
\]

which is actually a bifuzzy variable.

Definition 2. An \( n \)-dimensional bifuzzy vector is a function from a possibility space to the set of \( n \)-dimensional fuzzy vectors.

So, what is the relationship between bifuzzy variables and bifuzzy vectors? A theorem in Liu \(^{20}\) states that \((\eta_1, \eta_2, \cdots, \eta_n)\) is a fuzzy vector if and only if \( \eta_1, \eta_2, \cdots, \eta_n \) are fuzzy variables. According to this conclusion, we obtain the following result.

Theorem 1. The vector \((\xi_1, \xi_2, \cdots, \xi_n)\) is a bifuzzy vector if and only if \( \xi_1, \xi_2, \cdots, \xi_n \) are bifuzzy variables.

Proof. Write \( \xi = (\xi_1, \xi_2, \cdots, \xi_n) \). Suppose that the vector \( \xi \) is a bifuzzy vector on the possibility space \((\Theta, P(\Theta), \text{Pos})\). Then, for each \( \theta \in \Theta \), the vector \( \xi(\theta) \) is a fuzzy vector. It follows that \( \xi_1(\theta), \xi_2(\theta), \cdots, \xi_n(\theta) \) are fuzzy variables. Thus \( \xi_1, \xi_2, \cdots, \xi_n \) are bifuzzy variables.

Conversely, suppose that \( \xi_1, \xi_2, \cdots, \xi_n \) are bifuzzy variables defined on the possibility space \((\Theta, P(\Theta), \text{Pos})\). Then, for each \( \theta \in \Theta \), the variables \( \xi_1(\theta), \xi_2(\theta), \cdots, \xi_n(\theta) \)
are fuzzy variables. It follows that \( \boldsymbol{\xi}(\theta) = (\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta)) \) is a fuzzy vector. Thus \( \boldsymbol{\xi} \) is a bifuzzy vector. The theorem is proved.

**Theorem 2.** Let \( \boldsymbol{\xi} \) be an \( n \)-dimensional bifuzzy vector, and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. Then \( f(\boldsymbol{\xi}) \) is a bifuzzy variable.

**Proof.** Suppose that \( \boldsymbol{\xi} \) is defined on the possibility space \((\Theta, P(\Theta), \text{Pos})\). Thus for each \( \theta \in \Theta \), \( \xi(\theta) \) is a fuzzy vector and \( f(\xi(\theta)) \) is a fuzzy variable. Then \( f(\boldsymbol{\xi}) \) is a bifuzzy variable since it is a function from a possibility space to the set of fuzzy variables.

**Theorem 3.** Assume that \( \boldsymbol{\xi} \) is an \( n \)-dimensional bifuzzy vector defined on the possibility space \((\Theta, P(\Theta), \text{Pos})\). Then for any set \( B \) of \( \mathbb{R}^n \), we have
(a) the possibility \( \text{Pos}\{\xi(\theta) \in B\} \) is a fuzzy variable;
(b) the necessity \( \text{Nec}\{\xi(\theta) \in B\} \) is a fuzzy variable;
(c) the credibility \( \text{Cr}\{\xi(\theta) \in B\} \) is a fuzzy variable.

**Proof.** Since \( \text{Pos}\{\xi(\theta) \in B\} \), \( \text{Nec}\{\xi(\theta) \in B\} \) and \( \text{Cr}\{\xi(\theta) \in B\} \) are functions from the possibility space \((\Theta, P(\Theta), \text{Pos})\) to the set of real numbers (in fact, \([0, 1]\)), they are fuzzy variables.

**Theorem 4.** Let \( \boldsymbol{\xi} \) be an \( n \)-dimensional bifuzzy vector on the possibility space \((\Theta, P(\Theta), \text{Pos})\), and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. If the expected value \( E[f(\xi(\theta))] \) is finite for each \( \theta \in \Theta \), then \( E[f(\xi(\theta))] \) is a fuzzy variable.

**Proof.** Since the expected value \( E[f(\xi(\theta))] \) is a function from the possibility space \((\Theta, P(\Theta), \text{Pos})\) to the set of real numbers, it is a fuzzy variable.

Especially, if \( f(x) = x \), it follows from Theorem 4 that \( E[\xi(\theta)] \) is a fuzzy variable if \( E[\xi(\theta)] \) is finite for each \( \theta \).

4. Chance Measure

**Definition 3.** (Liu) Let \( \boldsymbol{\xi} \) be an \( n \)-dimensional bifuzzy vector defined on the possibility space \((\Theta, P(\Theta), \text{Pos})\), and \( B \) be a set of \( \mathbb{R}^n \). Then the chance of bifuzzy event \( \xi \in B \) is a function from \([0, 1]\) to \([0, 1]\), defined as

\[
\text{Ch}\{\xi \in B\}(\alpha) = \sup_{\text{Cr}(A) \geq \alpha} \inf_{\xi \in A} \text{Cr}\{\xi(\theta) \in B\}.
\]

**Remark 1.** It is easy to verify that \( 0 \leq \text{Ch}\{\xi \in B\}(\alpha) \leq 1 \) for any \( \alpha \in (0, 1] \).

**Theorem 5.** Let \( \boldsymbol{\xi} \) be an \( n \)-dimensional bifuzzy vector on the possibility space \((\Theta, P(\Theta), \text{Pos})\), and \( B \) be a set of \( \mathbb{R}^n \). For any given \( \alpha^* > 0.5 \), we write \( \beta^* = \text{Ch}\{\xi \in B\}(\alpha^*) \). Then we have

\[
\text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta^*\} \geq \alpha^*.
\]
Proof. From the definition of $\beta^*$, there exists a sequence $\{\beta_i\}$ with $\beta_i \uparrow \beta^*$ such that

$$\text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta_i\} \geq \alpha^*.$$ 

Thus we have

$$\lim_{i \to \infty} \text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta_i\} \geq \alpha^* > 0.5. \tag{8}$$

It is also easy to verify that

$$\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta_i\} \downarrow \{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta^*\}$$

as $i \to \infty$. It follows from (8) and the credibility semicontinuity law that

$$\text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta^*\} = \lim_{i \to \infty} \text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta_i\} \geq \alpha^*.$$

The proof is complete. \qed

Definition 4. Let $(\Theta, P(\Theta), \text{Pos})$ be a possibility space. Then the set

$$\Theta^+ = \{\theta \in \Theta \mid \text{Pos}\{\theta\} > 0\} \tag{9}$$

is called the kernel of the possibility space $(\Theta, P(\Theta), \text{Pos})$.

Theorem 6. Let $\xi$ be an $n$-dimensional bifuzzy vector on the possibility space $(\Theta, P(\Theta), \text{Pos})$, and $B$ be a set of $\mathbb{R}^n$. Then $\text{Ch}\{\xi \in B\}(\alpha)$ is a decreasing function of $\alpha$, and

$$\lim_{\alpha \to 0} \text{Ch}\{\xi \in B\}(\alpha) = \sup_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\}, \tag{10}$$

$$\text{Ch}\{\xi \in B\}(1) = \inf_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\} \tag{11}$$

where $\Theta^+$ is the kernel of $(\Theta, P(\Theta), \text{Pos})$.

Proof. For any given $\alpha_1$ and $\alpha_2$ with $0 < \alpha_1 < \alpha_2 \leq 1$, it is clear that

$$\text{Ch}\{\xi \in B\}(\alpha_1) = \sup_{\text{Cr}(A) \geq \alpha_1} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \in B\} \geq \sup_{\text{Cr}(A) \geq \alpha_2} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \in B\} = \text{Ch}\{\xi \in B\}(\alpha_2).$$

That is, $\text{Ch}\{\xi \in B\}(\alpha)$ is a decreasing function of $\alpha$.

Next we prove (10). On the one hand, for any $\alpha \in (0, 1]$, since $\text{Ch}\{\xi \in B\}(\alpha)$ is a decreasing function of $\alpha$, we have

$$\text{Ch}\{\xi \in B\}(\alpha) = \sup \{\beta \mid \text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta\} \geq \alpha\} \leq \sup \{\beta \mid \text{Cr}\{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) \in B\} \geq \beta\} > 0\} = \sup_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\}.$$
Letting \( \alpha \downarrow 0 \), we get
\[
\lim_{\alpha \downarrow 0} \text{Ch}\{\xi \in B\}(\alpha) \leq \sup_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\}.
\]  
(12)

On the other hand, for any \( \theta^* \in \Theta^+ \), we write \( \alpha^* = \text{Cr}\{\theta^*\} > 0 \). Since \( \text{Ch}\{\xi \in B\}(\alpha) \) is a decreasing function of \( \alpha \), we have
\[
\lim_{\alpha \downarrow 0} \text{Ch}\{\xi \in B\}(\alpha) \geq \text{Ch}\{\xi \in B\}(\alpha^*) = \sup_{\text{Cr}(A) \geq \alpha^*} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \in B\} \geq \text{Cr}\{\xi(\theta^*) \in B\}
\]
which states that
\[
\lim_{\alpha \downarrow 0} \text{Ch}\{\xi \in B\}(\alpha) \geq \sup_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\}.
\]  
(13)

It follows from (12) and (13) that (10) holds.

Finally, we prove (11). On the one hand, for any set \( A \) with \( \text{Cr}(A) = 1 \), it is clear that \( \Theta^+ \subseteq A \). Thus
\[
\text{Ch}\{\xi \in B\}(1) = \sup_{\text{Cr}(A) = 1} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \in B\} \leq \inf_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\}.
\]  
(14)

On the other hand, since \( \text{Cr}\{\Theta^+\} = 1 \), we have
\[
\text{Ch}\{\xi \in B\}(1) \geq \inf_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) \in B\}.
\]  
(15)

It follows from (14) and (15) that (11) holds. The theorem is proved. \( \square \)

Note that the chance of bifuzzy event is defined as not a number but a function. Now let us give some other definitions of chance measure.

**Definition 5.** Let \( \xi \) be an \( n \)-dimensional bifuzzy vector, and \( B \) be a set of \( \mathbb{R}^n \). For any real number \( \alpha \in (0, 1] \), the \( \alpha \)-chance of bifuzzy event \( \xi \in B \) is defined as the value of chance at \( \alpha \), i.e., \( \text{Ch}\{\xi \in B\}(\alpha) \), where \( \text{Ch} \) denotes the chance measure.

**Definition 6.** Let \( \xi \) be an \( n \)-dimensional bifuzzy vector, and \( B \) be a set of \( \mathbb{R}^n \). Then the equilibrium chance of bifuzzy event \( \xi \in B \) is defined as
\[
\text{Ch}^e\{\xi \in B\} = \sup_{0 < \alpha \leq 1} \{ \alpha \mid \text{Ch}\{\xi \in B\}(\alpha) \geq \alpha \}
\]  
(16)

where \( \text{Ch} \) denotes the chance measure.

**Definition 7.** Let \( \xi \) be an \( n \)-dimensional bifuzzy vector, and \( B \) be a set of \( \mathbb{R}^n \). Then the average chance of bifuzzy event \( \xi \in B \) is defined as
\[
\text{Ch}^a\{\xi \in B\} = \int_0^1 \text{Ch}\{\xi \in B\}(\alpha) \, d\alpha
\]  
(17)

where \( \text{Ch} \) denotes the chance measure.
5. Chance Distribution

Recall that a random variable has its probability distribution, and a fuzzy variable has its credibility distribution. Correspondingly, we initiate the definition of chance distribution for bifuzzy variable in this section.

**Definition 8.** Let \( \xi \) be a bifuzzy variable. The chance distribution \( \Phi : [-\infty, +\infty] \times (0, 1] \rightarrow [0, 1] \) of \( \xi \) is defined by \( \Phi(x; \alpha) = \text{Ch}\{\xi \leq x\}(\alpha) \).

**Theorem 7.** The chance distribution \( \Phi(x; \alpha) \) of bifuzzy variable is a decreasing and left-continuous function of \( \alpha \) for each fixed \( x \).

**Proof.** Let \( \Phi \) be the chance distribution of bifuzzy variable \( \xi \) defined on the possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \). For any given \( \alpha_1 \) and \( \alpha_2 \) with \( 0 < \alpha_1 < \alpha_2 \leq 1 \), it follows immediately from Theorem 6 that

\[
\Phi(x; \alpha_1) = \text{Ch}\{\xi \leq x\}(\alpha_1) \geq \text{Ch}\{\xi \leq x\}(\alpha_2) = \Phi(x; \alpha_2).
\]

Thus \( \Phi(x; \alpha) \) is a decreasing function of \( \alpha \).

Next we prove the left-continuity of \( \Phi(x; \alpha) \) with respect to \( \alpha \). Let \( \alpha \in (0, 1] \) and \( x \) be given. Since \( \Phi(x; \alpha) \) is a decreasing function of \( \alpha \), the limit \( \lim_{\alpha \downarrow \alpha'} \Phi(x; \alpha') \) exists and is not less than \( \Phi(x; \alpha) \). If the limit is equal to \( \Phi(x; \alpha) \), then the left-continuity is proved. Otherwise, there exists a sequence \( \{\alpha_i\} \) with \( \alpha_i \uparrow \alpha \) such that

\[
\lim_{i \to \infty} \Phi(x; \alpha_i) > \Phi(x; \alpha).
\]

Let \( z^* = (\lim_{i \to \infty} \Phi(x; \alpha_i) + \Phi(x; \alpha))/2 \). It is clear that

\[
\Phi(x; \alpha_i) > z^* > \Phi(x; \alpha)
\]

for all \( i \). It follows from \( \Phi(x; \alpha_i) > z^* \) that there exists \( A_i \in \mathcal{P}(\Theta) \) with \( \text{Cr}\{A_i\} \geq \alpha_i \) such that

\[
\inf_{\theta \in A_i} \text{Cr}\{\xi(\theta) \leq x\} > z^*
\]

for each \( i \). Now we define

\[
A^* = \bigcup_{i=1}^{\infty} A_i.
\]

It is clear that \( \text{Cr}\{A^*\} \geq \text{Cr}\{A_i\} \geq \alpha_i \). Letting \( i \to \infty \), we get \( \text{Cr}\{A^*\} \geq \alpha \). Thus

\[
\Phi(x; \alpha) \geq \inf_{\theta \in A^*} \text{Cr}\{\xi(\theta) \leq x\} \geq z^*.
\]

A contradiction proves the theorem. \( \square \)

**Theorem 8.** The chance distribution \( \Phi(x; \alpha) \) of bifuzzy variable is an increasing function of \( x \) for each fixed \( \alpha \), and

\[
\Phi(-\infty; \alpha) = 0, \Phi(+\infty; \alpha) = 1, \forall \alpha; \tag{18}
\]
\[ \lim_{x \to -\infty} \Phi(x; \alpha) \leq 0.5 \text{ if } \alpha > 0.5; \]  
\[ \lim_{x \to +\infty} \Phi(x; \alpha) \geq 0.5 \text{ if } \alpha < 0.5. \]  

Furthermore, if \( \alpha > 0.5 \) and \( \lim_{y \downarrow x} \Phi(y; \alpha) > 0.5 \) or \( \Phi(x; \alpha) \geq 0.5 \), then we have
\[ \lim_{y \downarrow x} \Phi(y; \alpha) = \Phi(x; \alpha). \]

**Proof.** Let \( \Phi \) be the chance distribution of bifuzzy variable \( \xi \) defined on the possibility space \((\Theta, P(\Theta), \text{Pos})\). For any given \( x_1 \) and \( x_2 \) with \(-\infty \leq x_1 < x_2 \leq +\infty\), it is clear that
\[ \Phi(x; \alpha) = \sup_{\text{Cr}(A) \geq \alpha} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \leq x_1\} \leq \sup_{\text{Cr}(A) \geq \alpha} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \leq x_2\} = \Phi(x_2; \alpha). \]

That is, the chance distribution \( \Phi(x; \alpha) \) is an increasing function of \( x \).

Now we prove (18). Since \( \xi(\theta) \) is a fuzzy variable for any \( \theta \in \Theta \), we have \( \text{Cr}(\xi(\theta) \leq -\infty) = 0 \) for any \( \theta \in \Theta \) according to Lemma 2. It follows that
\[ \Phi(-\infty; \alpha) = \sup \{ \beta \mid \text{Cr}(\theta \in \Theta \mid \text{Cr}(\xi(\theta) \leq -\infty) \geq \beta) \geq \alpha \} = 0. \]

Similarly, we have \( \text{Cr}(\xi(\theta) \leq +\infty) = 1 \) for any \( \theta \in \Theta \) according to Lemma 2. Thus
\[ \Phi(+\infty; \alpha) = \sup \{ \beta \mid \text{Cr}(\theta \in \Theta \mid \text{Cr}(\xi(\theta) \leq +\infty) \geq \beta) \geq \alpha \} = 1. \]

Next we prove (19) and (20). If \( \alpha > 0.5 \), since \( \sup_{\theta \in \Theta} \text{Pos}\{\theta\} = 1 \), there must exist an element \( \theta^* \in \Theta \) such that \( \text{Pos}\{\theta^*\} > 2 - 2\alpha \). Thus for any \( A \in P(\Theta) \) with \( \text{Cr}(A) \geq \alpha \), it is easy to verify that \( \theta^* \in A \). It follows from Lemma 2 that
\[ \lim_{x \to -\infty} \Phi(x; \alpha) = \lim_{x \to -\infty} \sup_{\text{Cr}(A) \geq \alpha} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \leq x\} \leq \lim_{x \to -\infty} \text{Cr}(\xi(\theta^*) \leq x) \leq 0.5. \]

When \( \alpha < 0.5 \), there exists an element \( \theta^* \in \Theta \) such that \( \text{Pos}\{\theta^*\} \geq 2\alpha \). Then \( \text{Cr}\{\theta^*\} \geq \text{Pos}\{\theta^*\}/2 \geq \alpha \). Thus we have
\[ \lim_{x \to +\infty} \Phi(x; \alpha) = \lim_{x \to +\infty} \sup_{\text{Cr}(A) \geq \alpha} \inf_{\theta \in A} \text{Cr}\{\xi(\theta) \leq x\} \geq \lim_{x \to +\infty} \text{Cr}(\xi(\theta^*) \leq x) \geq 0.5 \]

according to Lemma 2.

Finally, we prove (21). Suppose that \( \alpha > 0.5 \) and \( \lim_{y \downarrow x} \Phi(y; \alpha) > 0.5 \), and let \( \{x_i\} \) be an arbitrary sequence with \( x_i \downarrow x \) as \( i \to \infty \). Write
\[ \beta = \Phi(x; \alpha), \quad \beta_i = \Phi(x_i; \alpha), \quad i = 1, 2, \ldots \]
It is clear that \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta \). Thus the limit
\[ \rho = \lim_{i \to \infty} \beta_i = \lim_{y \downarrow x} \Phi(y; \alpha) > 0.5 \]

and \( \rho \geq \beta \). On the other hand, since \( \rho \leq \beta_i \) for each \( i \), we have
\[ \text{Cr}(\theta \in \Theta \mid \text{Cr}(\xi(\theta) \leq x_i) \geq \rho) \geq \text{Cr}(\theta \in \Theta \mid \text{Cr}(\xi(\theta) \leq x_i) \geq \beta_i) \geq \alpha. \]
Since $\alpha > 0.5$, we get
\[
\lim_{\rho \to \infty} \Cr_{\{\theta \in \Theta \mid \Cr{\xi(\theta) \leq x_i} \geq \rho\}} \geq \alpha > 0.5. \tag{22}
\]
It follows from $\rho > 0.5$ and the credibility semicontinuity law that
\[
\{\theta \in \Theta \mid \Cr{\xi(\theta) \leq x_i} \geq \rho\} \downarrow \{\theta \in \Theta \mid \Cr{\xi(\theta) \leq x} \geq \rho\}.
\]
Furthermore, it follows from (22) and the credibility semicontinuity law that
\[
\Cr{\{\theta \in \Theta \mid \Cr{\xi(\theta) \leq x} \geq \rho\}} = \lim_{\rho \to \infty} \Cr{\{\theta \in \Theta \mid \Cr{\xi(\theta) \leq x_i} \geq \rho\}} \geq \alpha
\]
which implies that $\rho \leq \beta$. Hence $\rho = \beta$ and (21) holds. Under the condition $\alpha > 0.5$ and $\Phi(x; \alpha) \geq 0.5$, if $\Phi(y; \alpha) \to \Phi(x; \alpha)$ as $y \downarrow x$, then (21) holds trivially. Otherwise, we have
\[
\lim_{y \downarrow x} \Phi(y; \alpha) > \Phi(x; \alpha) \geq 0.5
\]
which also implies (21). The theorem is proved. \hfill \qed

**Theorem 9.** Let $\xi$ be a bifuzzy variable. Then $\Ch{\xi \geq x}(\alpha)$ is
(a) a decreasing and left-continuous function of $\alpha$ for each fixed $x$;
(b) a decreasing function of $x$ for each fixed $\alpha$. Furthermore, if $\alpha > 0.5$ and $\Ch{\xi \geq x}(\alpha) \geq 0.5$ or $\lim_{y \downarrow x} \Ch{\xi \geq y}(\alpha) > 0.5$, then we have $\lim_{y \uparrow x} \Ch{\xi \geq y}(\alpha) = \Ch{\xi \geq x}(\alpha)$.

**Proof.** Like Theorems 7 and 8. \hfill \qed

**Definition 9.** The chance density function $\phi : \mathbb{R} \times (0, 1) \to [0, +\infty)$ of a bifuzzy variable $\xi$ is a function such that
\[
\Phi(x; \alpha) = \int_{-\infty}^{x} \phi(y; \alpha) \, dy \tag{23}
\]
holds for all $x \in [-\infty, +\infty]$ and $\alpha \in (0, 1)$, where $\Phi$ is the chance distribution of $\xi$.

**6. Expected Value Operator**

**Definition 10.** (Liu \cite{17}) Let $\xi$ be a bifuzzy variable on the possibility space $(\Theta, P(\Theta), \Pos)$. Then the expected value of $\xi$ is defined by
\[
E[\xi] = \int_{0}^{+\infty} \Cr{\theta \in \Theta \mid E[\xi(\theta)] \geq r} \, dr - \int_{-\infty}^{0} \Cr{\theta \in \Theta \mid E[\xi(\theta)] \leq r} \, dr \tag{24}
\]
provided that at least one of the two integrals is finite.

**Lemma 3.** (Liu and Liu \cite{18}) Let $\xi$ and $\eta$ be independent fuzzy variables with finite expected values. Then for any numbers $a$ and $b$, we have $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$. 

Theorem 10. Assume that $\xi$ and $\eta$ are bifuzzy variables with finite expected values. If (i) for each $\theta \in \Theta$, the fuzzy variables $\xi(\theta)$ and $\eta(\theta)$ are independent, and (ii) $E[\xi(\theta)]$ and $E[\eta(\theta)]$ are independent fuzzy variables, then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

(25)

Proof. For any $\theta \in \Theta$, since the fuzzy variables $\xi(\theta)$ and $\eta(\theta)$ are independent, it follows from Lemma 3 that

$$E[a\xi(\theta) + b\eta(\theta)] = aE[\xi(\theta)] + bE[\eta(\theta)].$$

In addition, since $E[\xi(\theta)]$ and $E[\eta(\theta)]$ are independent fuzzy variables, we have

$$E[a\xi + b\eta] = E\left[aE[\xi(\theta)] + bE[\eta(\theta)]\right] = aE\left[E[\xi(\theta)]\right] + bE\left[E[\eta(\theta)]\right] = aE[\xi] + bE[\eta].$$

The theorem is proved.

7. Variance, Covariance and Moments

In this section, we propose more parameters to measure bifuzzy variables involving variance, standard derivation, covariance, the $k$th moment and the $k$th central moment.

Definition 11. Let $\xi$ be a bifuzzy variable with finite expected value $e$. The variance of $\xi$ is defined as $V[\xi] = E[(\xi - e)^2]$.

Lemma 4. (Liu 19) Let $\xi$ be a fuzzy variable, and $p > 1$ be a number. Then we have $|E[\xi]|^p \leq E[|\xi|^p]$.

Theorem 11. Assume that $\xi$ is a bifuzzy variable whose expected value exists. Then we have $V[E[\xi(\theta)]] \leq V[\xi]$.

Proof. Denote the expected value of $\xi$ by $e$. Since $E[\xi(\theta)]$ is a fuzzy variable, it follows from Lemma 4 that

$$V[E[\xi(\theta)]] = E[(E[\xi(\theta)] - e)^2] \leq E[(\xi(\theta) - e)^2] = V[\xi].$$

The theorem is proved.

Theorem 12. Let $\xi$ be a bifuzzy variable with finite expected value $e$. Then $V[\xi] = 0$ if and only if $\text{Ch}\{\xi = e\}(1) = 1$.

Proof. Suppose that $\xi$ is defined on the possibility space $(\Theta, P(\Theta), \text{Pos})$. If $V[\xi] = 0$, then it follows from $V[\xi] = E[(\xi - e)^2]$ that

$$\int_0^{+\infty} \text{Cr}\{\theta \in \Theta \mid E[(\xi(\theta) - e)^2] \geq r\} \, dr = 0$$

if and only if $\text{Ch}\{\xi = e\}(1) = 1$. If $\text{Ch}\{\xi = e\}(1) = 1$, then $V[\xi] = E[(\xi - e)^2]$.
which implies \( \text{Cr}\{\theta \in \Theta | E[(\xi(\theta) - e)^2] \geq r} \} = 0 \) for any \( r > 0 \). Since the credibility measure is self dual, we have \( \text{Cr}\{\theta \in \Theta | E[(\xi(\theta) - e)^2] = 0}\} = 1 \). That is, for each \( \theta^* \in \Theta^+ \), the kernel of \((\Theta, P(\Theta), \text{Pos})\), we have \( E[(\xi(\theta^*) - e)^2] = 0 \), i.e.,

\[
\int_0^{+\infty} \text{Cr}\{(\xi(\theta^*) - e)^2 \geq r}\ dr = 0.
\]

Therefore, we have \( \text{Cr}\{(\xi(\theta^*) - e)^2 \geq r}\ = 0 \) for any \( r > 0 \) and \( \text{Cr}\{(\xi(\theta^*) - e)^2 = 0}\} = 1 \). That is, \( \text{Cr}\{\xi(\theta^*) = e\} = 1 \) for each \( \theta^* \in \Theta^+ \). Hence it follows from Theorem 6 that

\[
\text{Ch}\{\xi = e\}(1) = \inf_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) = e\} = 1.
\]

Conversely, if \( \text{Ch}\{\xi = e\}(1) = 1 \), then we have \( \inf_{\theta \in \Theta^+} \text{Cr}\{\xi(\theta) = e\} = 1 \), which states that \( \text{Cr}\{\xi(\theta^*) = e\} = 1 \) for any \( \theta^* \in \Theta^+ \). In other words, \( \text{Cr}\{(\xi(\theta^*) - e)^2 \geq r} \} = 0 \) for each \( r > 0 \). Thus

\[
E[(\xi(\theta^*) - e)^2] = \int_0^{+\infty} \text{Cr}\{(\xi(\theta^*) - e)^2 \geq r}\ dr = 0
\]

for each \( \theta^* \in \Theta^+ \). It follows that \( \text{Cr}\{\theta \in \Theta | E[(\xi(\theta) - e)^2] \geq r} \} = 0 \) for any \( r > 0 \). Hence

\[
V[\xi] = \int_0^{+\infty} \text{Cr}\{\theta \in \Theta | E[(\xi(\theta) - e)^2] \geq r}\ dr = 0.
\]

The theorem is proved. \( \square \)

**Definition 12.** The standard deviation of a bifuzzy variable is defined as the non-negative square root of its variance.

**Definition 13.** Let \( \xi \) and \( \eta \) be bifuzzy variables such that \( E[\xi] \) and \( E[\eta] \) are finite. Then the covariance of \( \xi \) and \( \eta \) is defined by \( \text{Cov}[\xi, \eta] = E[(\xi - E[\xi])(\eta - E[\eta])] \).

**Definition 14.** For any positive integer \( k \), the expected value \( E[\xi^k] \) is called the \( k \)th moment of bifuzzy variable \( \xi \).

**Definition 15.** For any positive integer \( k \), the expected value \( E[(\xi - E[\xi])^k] \) is called the \( k \)th central moment of bifuzzy variable \( \xi \).

8. Optimistic and Pessimistic Values

**Definition 16.** (Liu) Let \( \xi \) be a bifuzzy variable, and \( \gamma, \delta \in (0, 1] \). Then

\[
\xi_{\text{sup}}(\gamma, \delta) = \sup \{r | \text{Ch}\{\xi \geq r\} (\gamma) \geq \delta\}
\]

is called the \((\gamma, \delta)\)-optimistic value to \( \xi \), and

\[
\xi_{\text{inf}}(\gamma, \delta) = \inf \{r | \text{Ch}\{\xi \leq r\} (\gamma) \geq \delta\}
\]

is called the \((\gamma, \delta)\)-pessimistic value to \( \xi \).
Theorem 13. Let $\xi$ be a bifuzzy variable. Assume that $\xi_{\sup}(\gamma, \delta)$ and $\xi_{\inf}(\gamma, \delta)$ are the $(\gamma, \delta)$-optimistic and $(\gamma, \delta)$-pessimistic values to $\xi$, respectively. If $\gamma > 0.5$ and $\delta > 0.5$, then we have
\[ \text{Ch}\{\xi \leq \xi_{\inf}(\gamma, \delta)\}(\gamma) \geq \delta, \quad \text{Ch}\{\xi \geq \xi_{\sup}(\gamma, \delta)\}(\gamma) \geq \delta. \] (28)

Proof. It follows from the definition of $(\gamma, \delta)$-pessimistic value that there exists a sequence $\{x_i\}$ with $x_i \downarrow \xi_{\inf}(\gamma, \delta)$ such that $\text{Ch}\{\xi \leq x_i\}(\gamma) \geq \delta$. Thus we have
\[ \lim_{y \downarrow \xi_{\inf}(\gamma, \delta)} \text{Ch}\{\xi \leq y\}(\gamma) \geq \delta > 0.5. \]
It follows from $\gamma > 0.5$ and Theorem 8 that
\[ \text{Ch}\{\xi \leq \xi_{\inf}(\gamma, \delta)\}(\gamma) = \lim_{y \downarrow \xi_{\inf}(\gamma, \delta)} \text{Ch}\{\xi \leq y\}(\gamma) \geq \delta. \]
Similarly, it follows from the definition of $(\gamma, \delta)$-optimistic value that there exists a sequence $\{x_i\}$ with $x_i \uparrow \xi_{\sup}(\gamma, \delta)$ such that $\text{Ch}\{\xi \geq x_i\}(\gamma) \geq \delta$. Thus we have
\[ \lim_{y \uparrow \xi_{\sup}(\gamma, \delta)} \text{Ch}\{\xi \geq y\}(\gamma) \geq \delta > 0.5. \]
It follows from $\gamma > 0.5$ and Theorem 9 that
\[ \text{Ch}\{\xi \geq \xi_{\sup}(\gamma, \delta)\}(\gamma) = \lim_{y \uparrow \xi_{\sup}(\gamma, \delta)} \text{Ch}\{\xi \geq y\}(\gamma) \geq \delta. \]
The theorem is proved. \qed

Theorem 14. Let $\xi_{\sup}(\gamma, \delta)$ and $\xi_{\inf}(\gamma, \delta)$ be the $(\gamma, \delta)$-optimistic and $(\gamma, \delta)$-pessimistic values of bifuzzy variable $\xi$ on the possibility space $(\Theta, P(\Theta), \text{Pos})$, respectively. If $\gamma \leq 0.5$, then we have
\[ \xi_{\inf}(\gamma, \delta) \leq \xi_{\sup}(\gamma, \delta) + \delta_1; \] (29)
if $\gamma > 0.5$, then we have
\[ \xi_{\inf}(\gamma, \delta) + \delta_2 \geq \xi_{\sup}(\gamma, \delta), \] (30)
where
\[ \delta_1 = \sup_{\theta \in \Theta} \{\xi(\theta)_{\sup}(1 - \delta) - \xi(\theta)_{\inf}(1 - \delta)\}, \]
\[ \delta_2 = \sup_{\theta \in \Theta} \{\xi(\theta)_{\sup}(\delta) - \xi(\theta)_{\inf}(\delta)\} \]
and $\xi(\theta)_{\sup}(\delta)$ and $\xi(\theta)_{\inf}(\delta)$ are the $\delta$-optimistic and $\delta$-pessimistic values of fuzzy variable $\xi(\theta)$ for each $\theta$, respectively.

Proof. Assume that $\gamma \leq 0.5$. For any given $\varepsilon > 0$, we define
\[ \Theta_1 = \{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) > \xi_{\sup}(\gamma, \delta) + \varepsilon\} \geq \delta\}, \]
\[ \Theta_2 = \{\theta \in \Theta \mid \text{Cr}\{\xi(\theta) < \xi_{\inf}(\gamma, \delta) - \varepsilon\} \geq \delta\}. \]
It follows from the definitions of \( \xi_{\text{sup}}(\gamma, \delta) \) and \( \xi_{\text{inf}}(\gamma, \delta) \) that \( \text{Cr}\{\Theta_1\} < \gamma \) and \( \text{Cr}\{\Theta_2\} < \gamma \). Thus \( \text{Cr}\{\Theta_1\} + \text{Cr}\{\Theta_2\} < \gamma + \gamma \leq 1 \). This fact implies \( \Theta_1 \cup \Theta_2 \neq \emptyset \), and there exists \( \theta^* \notin \Theta_1 \cup \Theta_2 \) such that
\[
\text{Cr} \{ \xi(\theta^*) > \xi_{\text{sup}}(\gamma, \delta) + \varepsilon \} < \delta,
\]
\[
\text{Cr} \{ \xi(\theta^*) < \xi_{\text{inf}}(\gamma, \delta) - \varepsilon \} < \delta.
\]

Since the credibility measure is self dual, we have
\[
\text{Cr} \{ \xi(\theta^*) \leq \xi_{\text{sup}}(\gamma, \delta) + \varepsilon \} > 1 - \delta,
\]
\[
\text{Cr} \{ \xi(\theta^*) \geq \xi_{\text{inf}}(\gamma, \delta) - \varepsilon \} > 1 - \delta.
\]

It follows from the definitions of \( \xi(\theta^*)_{\text{sup}}(1 - \delta) \) and \( \xi(\theta^*)_{\text{inf}}(1 - \delta) \) (Liu 20) that
\[
\xi_{\text{sup}}(\gamma, \delta) + \varepsilon \geq \xi(\theta^*)_{\text{inf}}(1 - \delta),
\]
\[
\xi_{\text{inf}}(\gamma, \delta) - \varepsilon \leq \xi(\theta^*)_{\text{sup}}(1 - \delta)
\]
which implies that
\[
\xi_{\text{inf}}(\gamma, \delta) - \varepsilon - (\xi_{\text{sup}}(\gamma, \delta) + \varepsilon) \leq \xi(\theta^*)_{\text{sup}}(1 - \delta) - \xi(\theta^*)_{\text{inf}}(1 - \delta) \leq \delta_1.
\]

Letting \( \varepsilon \to 0 \), we obtain (29).

Next we prove the inequality (30). Assume that \( \gamma > 0.5 \). For any given \( \varepsilon > 0 \), we define
\[
\Theta_1 = \{ \theta \in \Theta \mid \text{Cr} \{ \xi(\theta) \geq \xi_{\text{sup}}(\gamma, \delta) - \varepsilon \} \geq \delta \},
\]
\[
\Theta_2 = \{ \theta \in \Theta \mid \text{Cr} \{ \xi(\theta) \leq \xi_{\text{inf}}(\gamma, \delta) + \varepsilon \} \geq \delta \}.
\]

It follows from the definitions of \( \xi_{\text{sup}}(\gamma, \delta) \) and \( \xi_{\text{inf}}(\gamma, \delta) \) that \( \text{Cr}\{\Theta_1\} \geq \gamma \) and \( \text{Cr}\{\Theta_2\} \geq \gamma \). Thus \( \text{Cr}\{\Theta_1\} + \text{Cr}\{\Theta_2\} \geq \gamma + \gamma \geq 1 \). This fact implies that \( \Theta_1 \cap \Theta_2 \neq \emptyset \), and there exists \( \theta^* \in \Theta_1 \cap \Theta_2 \) such that
\[
\text{Cr} \{ \xi(\theta^*) \geq \xi_{\text{sup}}(\gamma, \delta) - \varepsilon \} \geq \delta,
\]
\[
\text{Cr} \{ \xi(\theta^*) \leq \xi_{\text{inf}}(\gamma, \delta) + \varepsilon \} \geq \delta.
\]

It follows from the definitions of \( \xi(\theta^*)_{\text{sup}}(\gamma, \delta) \) and \( \xi(\theta^*)_{\text{inf}}(\gamma, \delta) \) (Liu 20) that
\[
\xi_{\text{sup}}(\gamma, \delta) - \varepsilon \leq \xi(\theta^*)_{\text{sup}}(\delta),
\]
\[
\xi_{\text{inf}}(\gamma, \delta) + \varepsilon \geq \xi(\theta^*)_{\text{inf}}(\delta)
\]
which implies that
\[
(\xi_{\text{sup}}(\gamma, \delta) - \varepsilon) - (\xi_{\text{inf}}(\gamma, \delta) + \varepsilon) \leq \xi(\theta^*)_{\text{sup}}(\delta) - \xi(\theta^*)_{\text{inf}}(\delta) \leq \delta_2.
\]

Letting \( \varepsilon \to 0 \), the inequality (30) is proved. \( \square \)
9. Bifuzzy Simulations

It is impossible to design an analytic algorithm to deal with general bifuzzy systems. In order to do that, we introduce some bifuzzy simulations by several examples for calculating expected value, finding critical value, and computing chance function in this section.

Example 5. Assume that $\xi$ is an $n$-dimensional bifuzzy vector defined on the possibility space $(\Theta, P(\Theta), \text{Pos})$, and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function. Then $f(\xi)$ is a bifuzzy variable whose expected value $E[f(\xi)]$ is

$$\int_0^{+\infty} \text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \geq r\}dr - \int_{-\infty}^0 \text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \leq r\}dr.$$

A bifuzzy simulation will be introduced to compute the expected value $E[f(\xi)]$. We randomly sample $\theta_k$ from $\Theta$ such that $\text{Pos}\{\theta_k\} \geq \varepsilon$, and denote $\nu_k = \text{Pos}\{\theta_k\}$ for $k = 1, 2, \ldots, N$, where $\varepsilon$ is a sufficiently small number. Then for any number $r \geq 0$, the credibility $\text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \geq r\}$ can be estimated by

$$\frac{1}{2} \left( \max_{1 \leq k \leq N} \{\nu_k \mid E[f(\xi(\theta_k))] \geq r\} + \min_{1 \leq k \leq N} \{1 - \nu_k \mid E[f(\xi(\theta_k))] < r\} \right)$$

and for any number $r < 0$, the credibility $\text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \leq r\}$ can be estimated by

$$\frac{1}{2} \left( \max_{1 \leq k \leq N} \{\nu_k \mid E[f(\xi(\theta_k))] \leq r\} + \min_{1 \leq k \leq N} \{1 - \nu_k \mid E[f(\xi(\theta_k))] > r\} \right)$$

provided that $N$ is sufficiently large, where $E[f(\xi(\theta_k))]$, $k = 1, 2, \ldots, N$ may be estimated by the fuzzy simulation. We describe the process as follows.

Step 1. Set $E = 0$.

Step 2. Randomly sample $\theta_k$ from $\Theta$ such that $\text{Pos}\{\theta_k\} \geq \varepsilon$, $k = 1, 2, \ldots, N$, where $\varepsilon$ is a sufficiently small number.

Step 3. Compute $E[f(\xi(\theta_k))]$ by fuzzy simulation for $k = 1, 2, \ldots, N$, respectively.

Step 4. Let $a = \min_{1 \leq k \leq N} E[f(\xi(\theta_k))]$ and $b = \max_{1 \leq k \leq N} E[f(\xi(\theta_k))]$.

Step 5. Randomly generate $r$ from $[a, b]$.

Step 6. If $r \geq 0$, then $E \leftarrow E + \text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \geq r\}$, where fuzzy simulation is used to calculate $\text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \geq r\}$.

Step 7. If $r < 0$, then $E \leftarrow E - \text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \leq r\}$, where fuzzy simulation is used to obtain $\text{Cr}\{\theta \in \Theta \mid E[f(\xi(\theta))] \leq r\}$.

Step 8. Repeat the fifth to seventh steps for $N$ times.

Step 9. $E[f(\xi)] = a \vee 0 + b \wedge 0 + E \cdot (b - a)/N$.

Now we employ the bifuzzy simulation above to compute the expected value

$$E[\sqrt{\xi_1 + \xi_2 + \xi_3 + \xi_4}]$$
where $\xi_i, i = 1, 2, 3, 4$ are bifuzzy variables defined as

\[
\begin{align*}
\xi_1 &= (\rho_1 - 1, \rho_1, \rho_1 + 1), \text{ with } \rho_1 = (1, 2, 3), \\
\xi_2 &= (\rho_2 - 1, \rho_2, \rho_2 + 1), \text{ with } \rho_2 = (2, 3, 4), \\
\xi_3 &= (\rho_3 - 1, \rho_3, \rho_3 + 1), \text{ with } \rho_3 = (3, 4, 5), \\
\xi_4 &= (\rho_4 - 1, \rho_4, \rho_4 + 1), \text{ with } \rho_4 = (4, 5, 6).
\end{align*}
\]

Different simulation numbers are taken to compute the expected value $E[\sqrt{\xi_1 + \xi_2 + \xi_3 + \xi_4}]$ and we obtain the results shown in Table 1, where $M$ is the number of fuzzy simulation and “Expect” means the expected value.

<table>
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<tr>
<th>$N$</th>
<th>$M$</th>
<th>Expect</th>
<th>Error</th>
<th>$N$</th>
<th>$M$</th>
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<th>Error</th>
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<td>3.7229</td>
<td>0.46%</td>
<td>10</td>
<td>1000</td>
<td>3.7317</td>
<td>0.70%</td>
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<tr>
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<td>11</td>
<td>1500</td>
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<td>0.21%</td>
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<tr>
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<td>3.7238</td>
<td>0.49%</td>
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</tr>
<tr>
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<td>0.41%</td>
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<td>0.37%</td>
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<td>6</td>
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<td>0.18%</td>
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<tr>
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<td>0.18%</td>
<td>18</td>
<td>5000</td>
<td>3.7058</td>
<td>0.00%</td>
</tr>
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</table>

In Table 1, we compare solutions when different parameters are taken. It appears that all the chances differ little from each other. In order to account for it, we present a parameter, called the relative error, i.e., $\frac{|\text{actual value} - \text{optimal value}|}{\text{optimal value}} \times 100\%$, where optimal value is the last one of all the twenty expected values obtained above since the solution is better when taking larger simulation numbers. The last column named by “error” in Table 1 is just this parameter. It follows from Table 1 that all the relative errors of the results obtained by performing over 1000 cycles are less than or equal to 0.70%. Thus the bifuzzy simulation is considered to be effective for this type of problems.

**Example 6.** Assume that $\xi$ is an $n$-dimensional bifuzzy vector defined on the possibility space $(\Theta, P(\Theta), \text{Pos})$, and $f : \mathbb{R}^n \to \mathbb{R}$ is a function. For any given confidence levels $\alpha$ and $\beta$, we need design a bifuzzy simulation to find the maximal value $\overline{f}$ such that

\[
\text{Ch} \left\{ f(\xi) \geq \overline{f} \right\} (\alpha) \geq \beta
\]

holds. That is, we must find the maximal value $\overline{f}$ such that

\[
\text{Cr} \left\{ \theta \in \Theta \mid \text{Cr} \left\{ f(\xi(\theta)) \geq \overline{f} \right\} \geq \beta \right\} \geq \alpha.
\]

We randomly generate $\theta_k$ from $\Theta$ such that $\text{Pos}\{\theta_k\} \geq \varepsilon$, and write $\nu_k = \text{Pos}\{\theta_k\}$, $k = 1, 2, \cdots, N$, respectively, where $\varepsilon$ is a sufficiently small number. For any number
θ_k, we search for the maximal value \( \bar{f}(\theta_k) \) such that \( \text{Cr}\{f(\xi(\theta_k)) \geq \bar{f}(\theta_k)\} \geq \beta \) by fuzzy simulation. For any number \( r \), we have

\[
L(r) = \frac{1}{2} \left( \max_{1 \leq k \leq N} \{\nu_k \mid \bar{f}(\theta_k) \geq r\} + \min_{1 \leq k \leq N} \{1 - \nu_k \mid \bar{f}(\theta_k) < r\} \right). \tag{31}
\]

It follows from monotonicity that we may employ bisection search to find the maximal value \( r \) such that \( L(r) \geq \alpha \). This value is an estimation of \( \bar{f} \). We summarize this process as follows.

**Step 1.** Generate \( \theta_k \) from \( \Theta \) such that \( \text{Pos}\{\theta_k\} \geq \varepsilon \), and write \( \nu_k = \text{Pos}\{\theta_k\} \), \( k = 1, 2, \cdots, N \), respectively, where \( \varepsilon \) is a sufficiently small number.

**Step 2.** Obtain the maximal value \( \bar{f}(\theta_k) \) such that \( \text{Cr}\{f(\xi(\theta_k)) \geq \bar{f}(\theta_k)\} \geq \beta \) by fuzzy simulation, \( k = 1, 2, \cdots, N \), respectively.

**Step 3.** Find the maximal value \( r \) such that \( L(r) \geq \alpha \) holds, where \( L(r) \) is defined by (31).

**Step 4.** Return \( r \).

Let us find the maximal value \( \bar{f} \) such that

\[
\text{Ch}\{\xi_1^2 + \xi_2^2 + \xi_3^2 \geq \bar{f}\} \geq 0.9,
\]

where \( \xi_1, \xi_2, \xi_3 \) are bifuzzy variables with the membership functions

\[
\begin{align*}
\mu_{\xi_1}(x) &= \exp[-|x - \rho_1|], \quad \text{with} \quad \mu_{\rho_1}(x) = [1 - (x - 1)^2] \lor 0, \\
\mu_{\xi_2}(x) &= \exp[-|x - \rho_2|], \quad \text{with} \quad \mu_{\rho_2}(x) = [1 - (x - 2)^2] \lor 0, \\
\mu_{\xi_3}(x) &= \exp[-|x - \rho_3|], \quad \text{with} \quad \mu_{\rho_3}(x) = [1 - (x - 3)^2] \lor 0.
\end{align*}
\]

Different simulation numbers are taken to compute the \((0.9,0.8)\)-optimistic value and we obtain the results shown in Table 2, where \( M \) is the number of fuzzy simulation and "Optimistic" means the optimistic value.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M )</th>
<th>Optimistic</th>
<th>error</th>
<th>( N )</th>
<th>( M )</th>
<th>Optimistic</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3000</td>
<td>3000</td>
<td>1.9977</td>
<td>5.35%</td>
<td>11</td>
<td>11000</td>
<td>11000</td>
</tr>
<tr>
<td>2</td>
<td>4000</td>
<td>4000</td>
<td>1.9659</td>
<td>3.68%</td>
<td>12</td>
<td>12000</td>
<td>12000</td>
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<tr>
<td>3</td>
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<td>5000</td>
<td>1.9261</td>
<td>1.58%</td>
<td>13</td>
<td>13000</td>
<td>13000</td>
</tr>
<tr>
<td>4</td>
<td>6000</td>
<td>6000</td>
<td>1.8929</td>
<td>0.17%</td>
<td>14</td>
<td>14000</td>
<td>14000</td>
</tr>
<tr>
<td>5</td>
<td>7000</td>
<td>7000</td>
<td>1.8577</td>
<td>2.03%</td>
<td>15</td>
<td>14000</td>
<td>11000</td>
</tr>
<tr>
<td>6</td>
<td>8000</td>
<td>8000</td>
<td>1.9206</td>
<td>1.29%</td>
<td>16</td>
<td>12000</td>
<td>14000</td>
</tr>
<tr>
<td>7</td>
<td>9000</td>
<td>9000</td>
<td>1.9047</td>
<td>0.45%</td>
<td>17</td>
<td>14000</td>
<td>13000</td>
</tr>
<tr>
<td>8</td>
<td>10000</td>
<td>9000</td>
<td>1.9047</td>
<td>0.45%</td>
<td>18</td>
<td>14000</td>
<td>14000</td>
</tr>
<tr>
<td>9</td>
<td>10000</td>
<td>10000</td>
<td>1.8888</td>
<td>0.39%</td>
<td>19</td>
<td>14000</td>
<td>15000</td>
</tr>
<tr>
<td>10</td>
<td>10000</td>
<td>13000</td>
<td>1.8903</td>
<td>0.31%</td>
<td>20</td>
<td>15000</td>
<td>15000</td>
</tr>
</tbody>
</table>

Similarly, we use "error", i.e., the relative error, to measure the difference between these computational results. From Table 2, we see that the maximal relative
error does not exceed 1.86% when taking the simulation numbers over 10000. Therefore, the bifuzzy simulation is efficient to solve this type of problems.

Example 7. Assume that ξ is an n-dimensional bifuzzy vector defined on the possibility space (Θ, P(Θ), Pos), and f : \(\mathbb{R}^n \rightarrow \mathbb{R}\) is a function. For any given confidence level \(\alpha\), we design a bifuzzy simulation to compute the \(\alpha\)-chance

\[ L = \text{Ch}\{f(\xi) \leq 0\}(\alpha). \]

Equivalently, we should estimate

\[ L = \sup \left\{ \beta \mid \text{Cr}\{\theta \in \Theta \mid \text{Cr}\{f(\xi(\theta)) \leq 0\} \geq \beta\} \geq \alpha \right\}. \quad (32) \]

We randomly generate \(\theta_k\) from \(\Theta\) such that Pos\{\(\theta_k\)\} \(\geq \varepsilon\), and write \(\nu_k = \text{Pos}\{\theta_k\}\), \(k = 1, 2, \ldots, N\), respectively, where \(\varepsilon\) is a sufficiently small number. For any number \(\theta_k\), by using fuzzy simulation, we can estimate the credibility \(h(\theta_k) = \text{Cr}\{f(\xi(\theta_k)) \leq 0\}\). For any number \(r\), we have

\[ L(r) = \frac{1}{2} \left( \max_{1 \leq k \leq N} \{ \nu_k \mid h(\theta_k) \geq r \} + \min_{1 \leq k \leq N} \{ 1 - \nu_k \mid h(\theta_k) < r \} \right). \quad (33) \]

It follows from monotonicity that we may employ bisection search to find the maximal value \(r\) such that \(L(r) \geq \alpha\). This value is an estimation of \(L\). We summarize this process as follows.

**Step 1.** Generate \(\theta_k\) from \(\Theta\) such that Pos\{\(\theta_k\)\} \(\geq \varepsilon\), and write \(\nu_k = \text{Pos}\{\theta_k\}\), \(k = 1, 2, \ldots, N\), respectively, where \(\varepsilon\) is a sufficiently small number.

**Step 2.** Compute \(h(\theta_k) = \text{Cr}\{f(\xi(\theta_k)) \leq 0\}\) by using fuzzy simulation, \(k = 1, 2, \ldots, N\), respectively.

**Step 3.** Find the maximal value \(r\) such that \(L(r) \geq \alpha\) holds, where \(L(r)\) is defined by (33).

**Step 4.** Return \(r\).

Now pay our attention to computing the chance

\[ \text{Ch}\{\xi_1 + \xi_2 + \xi_3 \geq 5\}(0.9) \]

where \(\xi_1, \xi_2, \xi_3\) are bifuzzy variables defined as

\[ \xi_1 = (\rho_1 - 1, \rho_1, \rho_1 + 1), \text{ with } \mu_{\rho_1}(x) = 1/[1 + (x - 1)^2], \]

\[ \xi_2 = (\rho_2 - 2, \rho_2, \rho_2 + 2), \text{ with } \mu_{\rho_2}(x) = 1/[1 + (x - 2)^2], \]

\[ \xi_3 = (\rho_3 - 3, \rho_3, \rho_3 + 3), \text{ with } \mu_{\rho_3}(x) = 1/[1 + (x - 3)^2]. \]

Different parameters of bifuzzy simulation are adopted to calculate the chance and corresponding results are shown in Table 3, where \(M\) is the number of fuzzy simulation and “Chance” is the value of 0.9-chance.

Similarly, we use “error”, i.e., the relative error, to measure the difference between these computational results. From Table 3, we see that the maximal relative error does not exceed 1.52% when taking the simulation numbers over 10000, which implies that the bifuzzy simulation is effective to solve this type of problems.
Table 3. Comparison of Solutions of Example 7.

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>Chance error</th>
<th>N</th>
<th>M</th>
<th>Chance error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3000</td>
<td>0.3492 0.11%</td>
<td>11</td>
<td>11000</td>
<td>0.3362 0.24%</td>
</tr>
<tr>
<td>2</td>
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<td>0.3298 1.67%</td>
<td>12</td>
<td>12000</td>
<td>0.3392 1.13%</td>
</tr>
<tr>
<td>3</td>
<td>5000</td>
<td>0.3209 4.32%</td>
<td>13</td>
<td>13000</td>
<td>0.3314 1.19%</td>
</tr>
<tr>
<td>4</td>
<td>6000</td>
<td>0.3369 0.45%</td>
<td>14</td>
<td>14000</td>
<td>0.3303 1.52%</td>
</tr>
<tr>
<td>5</td>
<td>7000</td>
<td>0.3389 1.04%</td>
<td>15</td>
<td>14000</td>
<td>0.3334 0.60%</td>
</tr>
<tr>
<td>6</td>
<td>8000</td>
<td>0.3336 0.54%</td>
<td>16</td>
<td>12000</td>
<td>0.3303 1.52%</td>
</tr>
<tr>
<td>7</td>
<td>9000</td>
<td>0.3378 0.72%</td>
<td>17</td>
<td>14000</td>
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<td>8</td>
<td>10000</td>
<td>0.3378 0.72%</td>
<td>18</td>
<td>14000</td>
<td>0.3303 1.52%</td>
</tr>
<tr>
<td>9</td>
<td>10000</td>
<td>0.3332 0.66%</td>
<td>19</td>
<td>14000</td>
<td>0.3354 0.00%</td>
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<td>0.3314 1.19%</td>
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<td>0.3354 0.00%</td>
</tr>
</tbody>
</table>

10. Conclusion

The mathematical properties of bifuzzy variable are investigated in the paper. We define the chance distribution for bifuzzy variable and discuss its properties. Following that, the linearity of expected value operator is proved. Furthermore, the relationship between the optimistic value and the pessimistic value of bifuzzy variable is discussed. Finally, bifuzzy simulations are suggested and illustrated by some numerical experiments which make it possible to solve optimization problems with bifuzzy parameters.

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References