Uncertain Distribution-Minimum Spanning Tree Problem

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This paper studies the minimum spanning tree problem on a graph with uncertain edge weights, which are formulated as uncertain variables. The concept of ideal uncertain minimum spanning tree (ideal UMST) is initiated by extending the definition of the uncertain α-minimum spanning tree to reflect the overall properties of the α-minimum spanning tree weights at any confidence level α ∈ [0, 1]. On the basis of this new concept, the definition of uncertain distribution-minimum spanning tree is proposed in three ways. Particularly, by considering the tail value at risk from the perspective of risk management, the notion of uncertain β-distribution-minimum spanning tree (β-distribution-UMST) is suggested. It is shown that the β-distribution-UMST is just the uncertain expected minimum spanning tree when β = 0. For any β ∈ [0, 1], this problem can be effectively solved via the proposed deterministic graph transformation-based approach with the aid of the β-distribution-path optimality condition. Furthermore, the proposed definitions and solutions are illustrated by some numerical examples.

Keywords: Minimum spanning tree; uncertain distribution-minimum spanning tree; ideal UMST; β-distribution-UMST; tail value at risk.

1. Introduction

Minimum spanning tree (MST) problem has usually been solved under the deterministic circumstance, which means that the edge weights associated to a graph are completely known and assumed to be crisp numbers. For this case, some well-known algorithms such as the Kruskal algorithm and the Prim algorithm can be employed to solve them in polynomial time. However, the edge weights are not always deterministic.
Considering nondeterministic factors may be encountered in the MST problems, two main streams of research have been developed in the past few decades, namely the stochastic MST problem (e.g. Refs. 8–11) and the fuzzy MST problem (e.g. Refs. 12–16), which model the nondeterministic phenomena as randomness and fuzziness, respectively.

The stochastic MST problem and fuzzy MST problem provide some appropriate frameworks to model the MST problem and also work well in some practice while dealing with the nondeterministic factors which can be interpreted as frequency and subjective indeterminacy, respectively. In spite of these successes, both randomness and fuzziness have been shown to be improper to describe some nondeterministic phenomena, particularly those arisen from the shortage of statistical data and thus involving some linguistic ambiguity and subjective estimation. As for the MST problem, due to some economic reasons or technical difficulties, we often lack observed data to get an accurate estimation on the edge weights of a graph (or a network) on which the problem is concerned. Consequently, some domain experts are invited to make an assessment and give some belief degrees about the uncertain edges weights. However, it has been illustrated by many studies (e.g. Ref. 18) that human beings usually overweight unlikely events, and thus the belief degrees given by these experts may have much larger variance than the real frequency. This makes the probability theory and the fuzzy set theory inappropriate to deal with the MST problem in this situation since both of them may lead to counterintuitive results. Fortunately, the uncertainty theory, proposed by Liu in 2007 and redefined in 2010 as a powerful tool to rationally deal with human beings’ belief degrees, provides an alternative appropriate framework to handle it.

Since the uncertainty theory provides a new powerful mathematical approach to handling nondeterministic phenomena involving subjective estimation, it has been applied to a great deal of uncertain network optimization problems (e.g. Refs. 20–23). Particularly, on the basis of the uncertainty theory, Peng and Li considered the uncertain minimum spanning tree (UMST) problem where the edge weights are assumed to be uncertain variables. Following that, Peng et al. reviewed the recent advances in uncertain network optimization, and introduced some general uncertain network optimization models based on uncertain programming. Recently, some other variants of the MST problem have also been studied within the framework of Liu’s uncertainty theory such as the uncertain inverse MST problem and the uncertain quadratic MST problem.

As for the UMST problem, after it was proposed by Peng and Li, Zhou et al. further discussed its path optimality conditions as well as effective solving methods. Since the edge weights are uncertain variables in the UMST problem, the weights of spanning trees are uncertain as well. Consequently, the definition of MST for the classical MST problem becomes useless for the UMST problem. Therefore, three typical types of uncertain minimum spanning trees, namely expected UMST, α-UMST, and most UMST, were proposed according to different decision criteria.
Uncertain Distribution-Minimum Spanning Tree Problem

by Peng and Li, and Zhou et al. The expected UMST compares the expected values of the weights of all possible spanning trees to determine the minimum one, whereas the $\alpha$-UMST considers the fractile of the uncertainty distributions of the spanning tree weights regarding a predetermined confidence level $\alpha$, and the most UMST is an equivalent counterpart of the $\alpha$-UMST. It should be noted that for a given graph, the found UMST may be different under different decision criteria, and all of these three types of UMST have their own limitations.

For the purpose of discussing the performance of possible UMSTs under different decision criteria, and putting forward a more persuasive alternative decision criterion, the notion of ideal UMST is initiated in this paper. Ideal UMST, which can be seen as an extension of the $\alpha$-UMST, reflects the overall properties of the $\alpha$-minimum spanning tree weights at any confidence level $\alpha \in [0,1]$. Usually, such an ideal UMST may not really exist and it is a suppositional concept, but it can provide a new natural and persuasive perspective to define a UMST. A UMST defined by referring to the ideal UMST is called the uncertain distribution-minimum spanning tree (distribution-UMST). Specifically, we propose the definition of distribution-UMST in three different ways and thus three types of distribution-UMST are defined accordingly, i.e., max-distribution-UMST, overall-distribution-UMST, and $\beta$-distribution-UMST. In particular, the $\beta$-distribution-UMST is suggested by considering the tail value at risk from the perspective of risk management. Some discussions concerning the path optimality condition as well as related solutions are also presented for the $\beta$-distribution-UMST problem.

The rest of this paper is organized as follows. Section 2 introduces some basic concepts concerning the uncertain variable as well as the classical MST problem. Section 3 proposes the notion of ideal UMST. In Sec. 4, three types of distribution-UMST are defined, and some properties of them are discussed as well. Following that, path optimality condition and solutions of the $\beta$-distribution-UMST problem are presented in Sec. 5. Finally, numerical examples are given in Sec. 6 for illustration.

2. Preliminaries

In this section, some fundamental concepts concerning the uncertain variable as well as the classical MST problem are introduced briefly. For more details, the reader may refer to Refs. 19,28.

2.1. Uncertain variable

Definition 1. (Liu) Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $\mathcal{M} : \mathcal{L} \rightarrow [0,1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}(\Gamma) = 1$ for the universal set $\Gamma$;

Axiom 2. (Duality Axiom) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event $\Lambda$;
**Axiom 3.** (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$, we have $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$.

Besides, the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

**Definition 2.** (Liu19) An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

**Definition 3.** (Liu19) Let $\xi$ be an uncertain variable. Its uncertainty distribution is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number $x$.

**Definition 4.** (Liu19) An uncertain variable $\xi$ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a, \\ (x - a)/(b - a), & \text{if } a \leq x \leq b, \\ 1, & \text{if } x \geq b, \end{cases}$$

(1)

denoted by $\mathcal{L}(a, b)$, where $a$ and $b$ are real numbers with $a < b$.

**Definition 5.** (Liu19) An uncertainty distribution $\Phi$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then the inverse function $\Phi^{-1}$ is called the inverse uncertainty distribution of $\xi$.

Note that the inverse uncertainty distribution $\Phi^{-1}$ is well defined on the open interval $(0, 1)$. If required, we may extend the domain to $[0, 1]$ via $\Phi^{-1}(0) = \lim_{\alpha \to 0^+} \Phi^{-1}(\alpha)$ and $\Phi^{-1}(1) = \lim_{\alpha \to 1^-} \Phi^{-1}(\alpha)$.

It is clear that the distribution of a linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is regular, and its inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$  (2)

**Definition 6.** (Liu20) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if $\mathcal{M}\{\bigwedge_{i=1}^{n} \xi_i \in B_i\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$ for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

**Theorem 1.** (Liu19) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively, and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous and strictly increasing function. Then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)).$$  (3)

**Definition 7.** (Liu19) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{-\infty}^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$$  (4)

provided that at least one of the two integrals is finite.
2.2. Classical minimum spanning tree problem

Following many earlier studies on the MST problem (e.g. Refs. 28,30), we formulate the classical MST as follows.

Let \( G = (V, E, \mathbf{w}) \) denote a connected graph, where \( V = \{v_1, v_2, \ldots, v_n\} \) is the vertex set, \( E = \{e_1, e_2, \ldots, e_m\} \) is the edge set, and \( \mathbf{w} = (w_1, w_2, \ldots, w_m)^T \) is the edge weight vector. A spanning tree \( T = (V, S) \) of \( G \) is a connected acyclic subgraph containing all vertices, where \( S \) is the set of edges contained in \( T \). For the sake of brevity, a spanning tree \( T \) is denoted by its edge set \( S \) in this paper. Then the classical MST problem is to find a spanning tree with the minimum total edge weight among all spanning trees of \( G \).

Definition 8. (Minimum Spanning Tree) Given a connected graph \( G = (V, E, \mathbf{w}) \), a spanning tree \( T^0 \) is said to be a minimum spanning tree if

\[
\sum_{e_i \in T^0} w_i \leq \sum_{e_j \in T} w_j
\]

holds for any spanning tree \( T \).

Furthermore, with regard to a spanning tree \( T \), the edges in the tree are called tree edges, whereas the edges not in the tree are called non-tree edges. For any non-tree edge \( e_j \), it is known that there must exist a unique path between the vertices of \( e_j \) such that it is composed of tree edges. This unique path is called the tree path of edge \( e_j \) and denoted by \( P_j \). On the basis of these concepts, the following path optimality condition of MST was proposed by Ahuja et al.\(^{30}\)

Theorem 2. (Ahuja et al.\(^{30}\)) Given a connected graph \( G = (V, E, \mathbf{w}) \), a spanning tree \( T^0 \) is a minimum spanning tree if and only if

\[
w_i - w_j \leq 0, \quad \forall e_j \in E \setminus T^0, e_i \in P_j
\]  

(5)

where \( E \setminus T^0 \) is the set of non-tree edges, and \( P_j \) is the tree path of edge \( e_j \).

3. Ideal Uncertain Minimum Spanning Tree

In this section, we first briefly review the uncertain minimum spanning tree problem, particularly three typical types of UMST. Then, the concept of ideal UMST is initiated. Some theorems concerning the weight distribution of the ideal UMST are presented as well.

3.1. Three typical types of UMST

In many real-world applications, since the edge weights cannot be precisely known due to the lack of observed data, the MST problem should be considered on a graph with uncertain edge weights. Consequently, the uncertain version of the classical MST problem, called the uncertain minimum spanning tree problem, which is formulated as follows, was investigated by Peng and Li\(^{24}\) and Zhou et al.\(^{28}\)

Let \( \tilde{G} = (V, E, \mathbf{\xi}) \) denote an uncertain graph consisting of the vertex set \( V = \{v_1, v_2, \ldots, v_n\} \), the edge set \( E = \{e_1, e_2, \ldots, e_m\} \), and the edge weight vector

...
\[ \xi = (\xi_1, \xi_2, \ldots, \xi_m)^T, \text{ where } \xi_i, i = 1, 2, \ldots, m, \text{ are independent uncertain variables with regular uncertainty distributions.} \]

Then, the weight of a spanning tree \( T \) of the graph \( \tilde{G} \) can be denoted as

\[ W(T, \xi) = \sum_{e_i \in T} \xi_i. \]  \hspace{1cm} (6)

Since \( \xi_i, i = 1, 2, \ldots, m, \) are uncertain variables, \( W(T, \xi) \) is an uncertain variable as well. Consequently, Definition 8 for the classical MST problem becomes useless for the uncertain version of the problem. In this case, the following three typical types of UMST, including expected UMST, \( \alpha \)-UMST, and most UMST, were proposed according to different decision criteria.

**Definition 9.** (Peng and Li) \(^{24}\) Expected UMST) Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \), a spanning tree \( T^0 \) is called an uncertain expected minimum spanning tree if

\[ E[W(T^0, \xi)] \leq E[W(T, \xi)] \]  \hspace{1cm} (7)

holds for any spanning tree \( T \), where \( E[W(T^0, \xi)] \) and \( E[W(T, \xi)] \) are the expected values of weights of spanning trees \( T^0 \) and \( T \), respectively.

**Definition 10.** (Peng and Li) \(^{24}\) \( \alpha \)-UMST) Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) and a predetermined confidence level \( \alpha \in (0, 1] \), a spanning tree \( T^0 \) is called an uncertain \( \alpha \)-minimum spanning tree if

\[ \min\{\omega \mid \mathcal{M}\{W(T^0, \xi) \leq \omega\} \geq \alpha\} \leq \min\{\omega \mid \mathcal{M}\{W(T, \xi) \leq \omega\} \geq \alpha\} \]  \hspace{1cm} (8)

holds for any spanning tree \( T \), and \( \min\{\omega \mid \mathcal{M}\{W(T^0, \xi) \leq \omega\} \geq \alpha\} \) is called the \( \alpha \)-minimum spanning tree weight, denoted by \( W_\alpha(T^0, \xi) \).

**Definition 11.** (Zhou et al.) \(^{28}\) Most UMST) Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) and a predetermined weight supremum \( x^* \), a spanning tree \( T^0 \) is called an uncertain most minimum spanning tree if

\[ \Phi_{T^0}(x^*) \geq \Phi_T(x^*) \]  \hspace{1cm} (9)

holds for any spanning tree \( T \), where \( \Phi_{T^0} \) and \( \Phi_T \) are the uncertainty distributions of tree weights \( W(T^0, \xi) \) and \( W(T, \xi) \), respectively.

Obviously, the expected-UMST takes the expected value of the tree weights into consideration while determining the minimum spanning tree, whereas the \( \alpha \)-UMST minimizes the critical value \( \omega \) at a given confidence level \( \alpha \) such that \( \mathcal{M}\{W(T^0, \xi) \leq \omega\} \geq \alpha \). Similarly to the \( \alpha \)-UMST, the most-UMST maximizes the chance that the tree weight does not exceed a predetermined supremum \( x^* \). The equivalent relationship between the \( \alpha \)-UMST and the most-UMST was revealed by Zhou et al.\(^{28}\) as well.
3.2. **Ideal UMST: Definition and properties**

As mentioned above, the expected-UMST only considers the mean or expected value of the tree weights. As a result, more characters concerning the distributions of the tree weights are not reflected under this decision criterion. For example, two trees with the same expected weight but different variances are treated as the same according to this criterion.

Meanwhile, as for the $\alpha$-UMST (or the most UMST, which is with equivalent relationship to the $\alpha$-UMST), it only reflects the local properties of the distributions at a given confidence level (or a predetermined supremum). Consequently, the minimum spanning tree found under this decision criterion may vary with the predetermined confidence level (or the predetermined supremum). Taking the examples given by Zhou et al.\textsuperscript{28} (see Examples 3 and 4 in Ref. 28) for instance, Fig. 1 shows the $\alpha$-minimum spanning tree weights (denoted by the solid line) of a given graph with respect to different confidence levels $\alpha$. The weight distributions of the related spanning trees are illustrated in the figure as well, which are denoted by dashed lines. It is shown that the cases $0 < \alpha \leq 0.54$, $0.54 < \alpha \leq 0.78$ and $0.78 < \alpha \leq 1$ are associated with different $\alpha$-UMSTs, respectively.

In short, all the three typical types of UMST have their own limitations. Therefore, an alternative decision criterion is suggested in this paper to result in a so-called **ideal uncertain minimum spanning tree**, which can be seen as a natural...
extension of the \( \alpha \)-UMST. To present the ideal UMST, we first define the ideal distribution of UMST as follows.

**Definition 12. (Ideal Distribution of UMST)** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \), for the set of all the spanning trees \( \{T^1, T^2, \ldots, T^k\} \) of \( \tilde{G} \), the function

\[
\Phi^*(x) = \max_{1 \leq j \leq k} \Phi_{T^j}(x)
\]  

is called the ideal distribution of the uncertain minimum spanning tree of \( \tilde{G} \), where \( \Phi_{T^j} \) are the uncertainty distributions of the weights of \( T^j \) for \( j = 1, 2, \ldots, k \), respectively.

It is clear that the ideal distribution of UMST is nothing but the envelope curve of the distributions of all the spanning trees of a given graph. Taking Fig. 1 for example, the solid line is just the ideal distribution, which also reflects the \( \alpha \)-minimum spanning tree weights with respect to different confidence levels.

**Definition 13. (Ideal UMST)** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \), a spanning tree (may not actually exist) with the ideal distribution of the uncertain minimum spanning tree \( \Phi^*(x) \) is called the ideal uncertain minimum spanning tree of \( \tilde{G} \), and denoted by \( T^* \).

Concerning the distribution of the weight of an ideal UMST (i.e., the ideal distribution of UMST), we have the following theorems.

**Theorem 3.** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi_i \) are with regular uncertainty distributions, its ideal distribution of the uncertain minimum spanning tree \( \Phi^*(x) \) is a continuous and strictly increasing function with respect to \( x \) on \( \{x|0 < \Phi^*(x) < 1\} \).

**Proof.** Following from the definition of ideal distribution of UMST, we just need to prove that \( \Phi^*(x) = \max_{1 \leq j \leq k} \Phi_{T^j}(x) \) is continuous and strictly increasing at each point \( x \) with \( 0 < \Phi^*(x) < 1 \), where \( \{T^j|1 \leq j \leq k\} \) denotes the set of all the spanning trees of \( \tilde{G} \).

For the given graph \( \tilde{G} \), there are at most \( C_{n-1}^m \) spanning trees. That is, the number of possible spanning trees is a limited and fixed value. Thus, we can denote the set of all the spanning trees by listing them as \( \{T^1, T^2, \ldots, T^k\} \).

It follows from Theorem 1 that for any spanning tree \( T^j \), the tree weight \( W(T^j, \xi) \) is also an uncertain variable with a regular uncertainty distribution, denoted as \( \Phi_{T^j}(x) \). Then \( \Phi_{T^j}(x) \) and \( \Phi_{T^j}^{-1}(\alpha) \) are both continuous and strictly increasing on the domains \( \{x|0 < \Phi_{T^j}(x) < 1\} \) and \( \{\alpha|0 \leq \alpha \leq 1\} \), respectively.

We define that

\[
x^L = \min_{1 \leq j \leq k} \Phi_{T^j}^{-1}(0) = \min_{1 \leq j \leq k} \lim_{\alpha \to 0^+} \Phi_{T^j}^{-1}(\alpha)
\]  

(11)
\[ x^U = \min_{1 \leq j \leq k} \Phi^{-1}_{T_j}(1) = \min_{1 \leq j \leq k} \lim_{\alpha \to 1} \Phi^{-1}_{T_j}(\alpha). \tag{12} \]

It is easy to verify that for arbitrary \( x \in (x^L, x^U) \) (i.e., \( 0 < \max_{1 \leq j \leq k} \Phi_{T_j}(x) < 1 \), \( \Phi^*(x) = \max_{1 \leq j \leq k} \Phi_{T_j}(x) \) exists and is unique, which means \( \Phi^*(x) \) is a function map of \( x \).

Now let us prove \( \Phi^*(x) \) is continuous and strictly increasing. According to the arithmetic rule of limitation of compound function, we have
\[
\lim_{x \to x^*} \Phi^*(x) = \lim_{x \to x^*} \max_{1 \leq j \leq k} \Phi_{T_j}(x) = \max_{1 \leq j \leq k} \lim_{x \to x^*} \Phi_{T_j}(x) = \Phi^*(x^*). \tag{13} \]

That means \( \Phi^*(x) \) is continuous.

For arbitrary \( x^L < x_1 < x_2 < x^U \), we get
\[
\Phi^*(x_1) = \max_{1 \leq j \leq k} \Phi_{T_j}(x_1) \quad \text{and} \quad \Phi^*(x_2) = \max_{1 \leq j \leq k} \Phi_{T_j}(x_2). \tag{14} \]

For each spanning tree \( T_j \), \( \Phi_{T_j}(x) \) is continuous and strictly increasing on the domain \( \{x|0 < \Phi_{T_j}(x) < 1\} \), and \( \Phi_{T_j}(x) = 0 \) if \( x \leq \lim_{\alpha \to 0^+} \Phi_{T_j}^{-1}(\alpha) \) and \( \Phi_{T_j}(x) = 1 \) if \( x \geq \lim_{\alpha \to 1^-} \Phi_{T_j}^{-1}(\alpha) \). Since \( x^L < x_1 < x_2 < x^U \), there must exist at least one spanning tree \( T^0 \) such that \( \lim_{\alpha \to 0^+} \Phi_{T^0}^{-1}(\alpha) < x_1 < x_2 < \lim_{\alpha \to 1^-} \Phi_{T^0}^{-1}(\alpha) \). Then we have \( \Phi_{T^0}(x_1) < \Phi_{T^0}(x_2) \). For any other spanning tree \( T_j \neq T^0 \), \( \Phi_{T_j}(x_1) \leq \Phi_{T_j}(x_2) \) holds. Therefore, we have
\[
\max_{1 \leq j \leq k} \Phi_{T_j}(x_1) < \max_{1 \leq j \leq k} \Phi_{T_j}(x_2). \tag{15} \]

That is, \( \Phi^*(x_1) < \Phi^*(x_2) \). Hence \( \Phi^*(x) \) is strictly increasing with respect to \( x \in (x^L, x^U) \), i.e., \( 0 < \Phi^*(x) < 1 \).

According to Theorem 3, for a graph with uncertain edge weights which have regular distributions, the distribution of its ideal UMST is continuous and strictly increasing, and thus its inverse function exists. By utilizing its inverse uncertainty distribution, an equivalent formulation of the ideal distribution of UMST can be obtained as follows.

**Theorem 4.** (Equivalent Formulation of Ideal Distribution) Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i \), \( i = 1, 2, \ldots, m \), respectively, the set of all the spanning trees in \( \tilde{G} \) is denoted as \( \{T^1, T^2, \ldots, T^k\} \). For the inverse function of the ideal distribution of the uncertain minimum spanning tree of \( \tilde{G} \), called the inverse ideal distribution of UMST, we have
\[
(\Phi^*)^{-1}(\alpha) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha), \quad 0 \leq \alpha \leq 1, \tag{16} \]

where \( \Phi_{T_j}^{-1} \) denote the inverse distributions of the weights of \( T_j \) for \( j = 1, 2, \ldots, k \), respectively.
Proof. Since $\xi_i$, $i = 1, 2, \ldots, m$, are independent uncertain variables with regular uncertainty distributions, the distribution $\Phi_{T_j}(x)$ and inverse distribution $\Phi_{T_j}^{-1}(\alpha)$ are both continuous and strictly increasing on the domains $\{x | 0 < \Phi_{T_j}(x) < 1\}$ and $\{\alpha | 0 \leq \alpha \leq 1\}$, respectively. Furthermore, it follows from Theorem 3 that $(\Phi^*)^{-1}(\alpha)$ exists and is unique on the domain $0 < \alpha < 1$, i.e., $x \in (x^L, x^U)$.

For arbitrary $x^0 \in (x^L, x^U)$, according to the definition of ideal distribution, we have

$$\Phi^*(x^0) = \max_{1 \leq j \leq k} \Phi_{T_j}(x^0). \quad (17)$$

Then denote

$$\alpha_0 = \Phi^*(x^0) \quad \text{and} \quad \alpha_{T_j} = \Phi_{T_j}(x^0). \quad (18)$$

We have

$$\alpha_0 = \max_{1 \leq j \leq k} \alpha_{T_j} \quad (19)$$

and

$$(\Phi^*)^{-1}(\alpha_0) = \Phi_{T_j}^{-1}(\alpha_{T_j}) = x^0. \quad (20)$$

Since $\Phi_{T_j}^{-1}$ is strictly increasing, we obtain that for any spanning tree $T_j$,

$$\Phi_{T_j}^{-1}(\alpha_0) \geq \Phi_{T_j}^{-1}(\alpha_{T_j}) = (\Phi^*)^{-1}(\alpha_0). \quad (21)$$

That is

$$(\Phi^*)^{-1}(\alpha_0) = \min_{1 \leq j \leq k} \Phi_{T_j}^{-1}(\alpha_0), \quad 0 < \alpha_0 < 1, \quad (22)$$

which means Eq. (16) holds if $0 < \alpha < 1$.

By extending the domain of $\alpha$ to $[0, 1]$ via

$$\Phi_{T_j}^{-1}(0) = \lim_{\alpha \to 0^+} \Phi_{T_j}^{-1}(\alpha), \quad \Phi_{T_j}^{-1}(1) = \lim_{\alpha \to 1^-} \Phi_{T_j}^{-1}(\alpha), \quad (23)$$

and

$$(\Phi^*)^{-1}(0) = \lim_{\alpha \to 0^+} (\Phi^*)^{-1}(\alpha), \quad (\Phi^*)^{-1}(1) = \lim_{\alpha \to 1^-} (\Phi^*)^{-1}(\alpha), \quad (24)$$

Eq. (16) can be similarly verified. $\square$

Following from the definitions of ideal distribution and ideal UMST and Theorem 4, it is easy to find that the ideal UMST is a spanning tree such that it is always the most UMST for any predetermined weight supremum, and also the $\alpha$-UMST for any given confidence level. Consequently, such an ideal UMST may not really exist for most cases, and it is actually a suppositional concept. However, it provides a new perspective to define an uncertain minimum spanning tree. On the basis of this novel concept, some new decision criteria to find a UMST are discussed in the remainder of this paper.
4. Uncertain Distribution-Minimum Spanning Tree

In this section, we propose the definition of uncertain distribution-minimum spanning tree based on the notion of ideal UMST in three ways, which imply three different decision criteria for determining the UMST, respectively. Accordingly, three types of distribution-UMST are obtained, namely max-distribution-UMST, overall-distribution-UMST, and \( \beta \)-distribution-UMST. Some properties of the proposed distribution-UMSTs are discussed as well.

4.1. Max-distribution-UMST

Taking the ideal UMST as a reference to define a UMST, a natural way is to find a spanning tree such that it is the closest one to the ideal UMST among all spanning trees. Particularly, to measure the degree of “closeness” between a given spanning tree and the ideal UMST by the maximum distance between their weight distributions, a so-called max-distribution-UMST is suggested as follows.

**Definition 14.** (Max-Distribution-UMST) Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \), a spanning tree \( T_0 \) is called an uncertain max-distribution-minimum spanning tree if

\[
\max_x (\Phi^* - \Phi_{T_0}(x)) \leq \max_x (\Phi^* - \Phi_T(x))
\]

holds for any spanning tree \( T \), where \( \Phi_{T_0}, \Phi_T \) and \( \Phi^* \) are the weight distributions of spanning trees \( T_0 \), \( T \) and the ideal UMST, respectively.

In this definition, the degree of “closeness” between the spanning tree \( T \) and the ideal UMST \( T^* \) is measured by the maximum distance between their weight distributions \( \Phi^*(x) \) and \( \Phi_T(x) \) with respect to an arbitrary \( x \) in the whole domain as shown in Fig. 2.

![Fig. 2. The degree of closeness measured by the maximum distance.](image)

It is easy to see that this way is followed from the idea of the worst-case analysis, and the result may be influenced by some individual extreme points. To overcome this limitation, a natural extension of this definition is to consider the whole area surrounded by the weight distributions \( \Phi^*(x) \) and \( \Phi_T(x) \) as shown in Fig. 3.
4.2. Overall-distribution-UMST

By measuring the degree of “closeness” between a spanning tree $T$ and the ideal UMST $T^*$ by the area surrounded by $\Phi^*(x)$ and $\Phi_T(x)$, a corresponding new type of distribution-UMST, called the overall-distribution-UMST, is proposed as follows.

**Definition 15.** (Overall-Distribution-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, a spanning tree $T^0$ is called an uncertain overall-distribution-minimum spanning tree if

$$
\int_0^{\infty} (\Phi^*(x) - \Phi_{T^0}(x))dx \leq \int_0^{\infty} (\Phi^*(x) - \Phi_T(x))dx
$$

(26)

holds for any spanning tree $T$, where $\Phi_{T^0}$, $\Phi_T$ and $\Phi^*$ are the weight distributions of $T^0$, $T$ and the ideal UMST, respectively.

**Remark 1.** It is easy to verify that inequality (26) can be formulated in an equivalent form as follows,

$$
\int_0^{\infty} (1 - \Phi_{T^0}(x))dx \leq \int_0^{\infty} (1 - \Phi_T(x))dx.
$$

(27)

Since the weight of a spanning tree $W(T, \xi)$ cannot take negative values (i.e., $\Phi_T(x) = 0$ for any $x \leq 0$), the left and right items in inequality (27) are just the expected values of the weights of spanning tree $T^0$ and $T$, respectively, according to the definition of expected value of an uncertain variable (see Definition 7). That is, $\int_0^{\infty} (1 - \Phi_{T^0}(x))dx = E[W(T^0, \xi)]$ and $\int_0^{\infty} (1 - \Phi_T(x))dx = E[W(T, \xi)]$. Therefore, we have the following theorem.

**Theorem 5.** (Equivalent Definition I of Overall-Distribution-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$, a spanning tree $T^0$ is an uncertain overall-distribution-minimum spanning tree if and only if it is an uncertain expected minimum spanning tree.

**Proof.** It follows from Remark 1 and the definition of expected UMST (Definition 9) immediately.
According to Theorem 4, if the edge weights of the graph $\tilde{G}$ are all independent uncertain variables with regular uncertainty distributions, its ideal distribution of UMST can be formulated in an equivalent form by using its inverse distribution. Consequently, an equivalent definition of the overall-distribution-UMST can be obtained for the case of $\tilde{G}$ with uncertain edge weights having regular uncertainty distributions as follows.

**Theorem 6.** (Equivalent Definition II of Overall-Distribution-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ where $\xi_i$ are with regular uncertainty distributions, a spanning tree $T^0$ is an uncertain overall-distribution-minimum spanning tree if and only if

$$\int_0^1 (\Phi_{T^0}^{-1}(\alpha) - (\Phi^*)^{-1}(\alpha))d\alpha \leq \int_0^1 (\Phi_T^{-1}(\alpha) - (\Phi^*)^{-1}(\alpha))d\alpha$$

holds for any spanning tree $T$, where $\Phi_{T^0}^{-1}$, $\Phi_T^{-1}$ and $(\Phi^*)^{-1}$ are the inverse distributions of the weights of $T^0$, $T$ and the ideal UMST, respectively.

**Proof.** Since all the edge weights $\xi_i$ are independent uncertain variables with regular uncertainty distributions, $\Phi_T(x)$ and $\Phi_T^{-1}(\alpha)$ are both continuous and strictly increasing on the domains $\{x\} | 0 < \Phi_T(x) < 1 \}$ and $\{\alpha\} | 0 \leq \alpha \leq 1 \}$, respectively. It follows from Theorems 3 and 4 that both $\Phi^*(x)$ and $(\Phi^*)^{-1}(\alpha)$ are also continuous and strictly increasing on the domains $\{x\} | 0 < \Phi^*(x) < 1 \}$ and $\{\alpha\} | 0 \leq \alpha \leq 1 \}$, respectively. Therefore, an equivalent form of inequality (26) can be immediately obtained as inequality (28), which calculates the area surrounded by $\Phi^*(x)$ and $\Phi_T(x)$ by integration with respect to $\alpha$ (see Fig. 3).

It is clear that the overall-distribution-UMST takes overall properties of the weight distributions into consideration. In other words, it considers all of the possible confidence levels $\alpha \in [0, 1]$ to measure the degree of “closeness” between $\Phi_T^{-1}(\alpha)$ and $(\Phi^*)^{-1}(\alpha)$ according to Theorem 6.

However, in reality, we scarcely care about the cases where the confidence level $\alpha$ is relatively low. Conversely, we always prefer to concern the situation that the confidence level $\alpha$ is above a certain level. From the perspective of risk management, the tail value at risk (also known as conditional value at risk) is usually being paid more concerns. Therefore, we further extend the definition of overall-distribution-UMST to a more general form by introducing a predetermined confidence level $\beta$ in the following.

### 4.3. $\beta$-distribution-UMST

For a given confidence level $\beta$, if the degree of “closeness” between a spanning tree $T$ and the ideal UMST $T^*$ is measured by the area surrounded by $\Phi_T^{-1}(\alpha)$ and
on condition that $\alpha \geq \beta$, a so-called $\beta$-distribution-UMST is suggested as follows.

**Definition 16.** ($\beta$-Distribution-UMST) Given a connected uncertain graph $\tilde{G} = (V, E, \xi)$ where $\xi_{i}$ are with regular uncertainty distributions, and a predetermined confidence level $\beta \in [0, 1]$, a spanning tree $T^{0}$ is an uncertain $\beta$-distribution-minimum spanning tree if

$$\int_{\beta}^{1} (\Phi_{T^{0}}^{-1}(\alpha) - (\Phi^*)^{-1}(\alpha))d\alpha \leq \int_{\beta}^{1} (\Phi_{T}^{-1}(\alpha) - (\Phi^*)^{-1}(\alpha))d\alpha$$

(29)

holds for any spanning tree $T$, where $\Phi_{T^{0}}^{-1}$, $\Phi_{T}^{-1}$ and $(\Phi^*)^{-1}$ are the inverse distributions of the weights of $T^{0}$, $T$ and the ideal UMST, respectively.

The geometric interpretation of the definition of $\beta$-distribution-UMST are illustrated in Fig. 4. Comparing it with Fig. 3, it is easy to see that the overall-distribution-UMST utilizes the whole area surrounded by the weight distribution of $T$ and the ideal distribution to measure the degree of “closeness” between a spanning tree $T$ and the ideal UMST $T^*$, whereas the $\beta$-distribution-UMST only considers the area satisfying $\alpha \geq \beta$.

![Fig. 4. Uncertain $\beta$-distribution minimum spanning tree.](image)

**Remark 2.** The definition of $\beta$-distribution-UMST is a natural extension of the overall-distribution-UMST since the latter is just a special case of the former. If $\beta = 0$, the $\beta$-distribution-UMST is an overall-distribution-UMST, and also an expected UMST according to Theorem 5.

**Remark 3.** It is easy to verify that inequality (29) in the definition of $\beta$-distribution-UMST can be formulated by the following equivalent form,

$$\int_{\beta}^{1} \Phi_{T^{0}}^{-1}(\alpha)d\alpha \leq \int_{\beta}^{1} \Phi_{T}^{-1}(\alpha)d\alpha .$$

(30)

For convenience, $\int_{\beta}^{1} \Phi_{T}^{-1}(\alpha)d\alpha$ is called the $\beta$-distribution weight of the spanning tree $T$. 
In order to interpret the idea behind the notion of $\beta$-distribution-UMST more clearly from the perspective of risk management, the concept of tail value at risk under uncertainty proposed by Peng\textsuperscript{31} is introduced as follows.

**Definition 17.** (Peng\textsuperscript{31}) Let $\xi$ be an uncertain variable, and $\beta \in (0,1)$ the risk confidence level. Then the tail value at risk of $\xi$ is defined by

$$
\xi_{\text{TVaR}}(\beta) = \frac{1}{1-\beta} \int_0^1 \inf \{ x \mid \mathcal{M}(\xi \leq x) \geq \alpha \} \, d\alpha.
$$

(31)

Based on the concept of tail value at risk, an equivalent definition of the $\beta$-distribution-UMST is obtained as follows.

**Theorem 7.** (Equivalent Definition I of $\beta$-Distribution-UMST) Given a connected uncertain graph $\tilde{G} = (V,E,\xi)$ where $\xi_i$ are with regular uncertainty distributions, and a predetermined confidence level $\beta \in (0,1)$, a spanning tree $T^0$ is an uncertain $\beta$-distribution-minimum spanning tree if and only if

$$
W(T^0,\xi)_{\text{TVaR}}(\beta) \leq W(T,\xi)_{\text{TVaR}}(\beta)
$$

holds for any spanning tree $T$, where $W(T^0,\xi)_{\text{TVaR}}(\beta)$ and $W(T,\xi)_{\text{TVaR}}(\beta)$ are the tail values at risk of the weights of spanning trees $T^0$ and $T$, respectively.

**Proof.** Since all the edge weights $\xi_i$ are independent uncertain variables with regular uncertainty distributions, $\Phi_T(x)$ and $\Phi_T^{-1}(\alpha)$ are both continuous and strictly increasing on the domains $\{x \mid 0 < \Phi_T(x) < 1\}$ and $\{\alpha \mid 0 \leq \alpha \leq 1\}$, respectively. Therefore, we have

$$
W(T^0,\xi)_{\text{TVaR}}(\beta) \leq W(T,\xi)_{\text{TVaR}}(\beta)
$$

$\Leftrightarrow \frac{1}{1-\beta} \int_0^1 \inf \{ x \mid \mathcal{M}(W(T^0,\xi) \leq x) \geq \alpha \} \, d\alpha
\leq \frac{1}{1-\beta} \int_0^1 \inf \{ x \mid \mathcal{M}(W(T,\xi) \leq x) \geq \alpha \} \, d\alpha
\Leftrightarrow \int_0^1 \Phi_T^{-1}(\alpha) \, d\alpha \leq \int_0^1 \Phi_T^{-1}(\alpha) \, d\alpha.
$$

According to Theorem 7, the definition of $\beta$-Distribution-UMST equals to taking the tail value at risk of the weight distribution of a spanning tree as the decision criterion to determine the minimum spanning tree. The solutions of the $\beta$-distribution-UMST are discussed in the following sections.

5. $\beta$-Distribution-Path Optimality Condition

The path optimality condition formulated by Theorem 2, which is a necessary and sufficient condition of the classical minimum spanning tree, provides a useful property for solving the classical MST problem. Following from this idea, in
this section, some equivalent definitions of the β-distribution-UMST as well as the β-distribution-path optimality condition are presented. Subsequently, an effective method to find the β-distribution-UMST is given on the basis of the proposed β-distribution-path optimality condition.

First of all, concerning the weight of a spanning tree, we have the following result.

**Theorem 8.** Given a connected uncertain graph \( \tilde{G} = (V,E,\xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i \), \( i = 1,2,\ldots,m \), respectively, and a predetermined confidence level \( \beta \in [0,1] \), for any spanning tree \( T \), we have

\[
\int_{\beta}^{1} \Phi_T^{-1}(\alpha) d\alpha = \sum_{e_i \in T} \int_{\beta}^{1} \Phi_i^{-1}(\alpha) d\alpha, \tag{33}
\]

where \( \Phi_T^{-1} \) is the inverse distribution of the weight of \( T \).

**Proof.** Since \( W(T,\xi) = \sum_{e_i \in T} \xi_i \) and \( \xi_i \) are independent uncertain variables, according to Theorem 1, we have

\[
\Phi_T^{-1}(\alpha) = \sum_{e_i \in T} \Phi_i^{-1}(\alpha), \quad \alpha \in [0,1]. \tag{34}
\]

Then Eq. (33) can be immediately obtained. \( \square \)

Based on Theorem 8, we can further get an equivalent definition of the β-Distribution-UMST by utilizing the integration of the inverse uncertainty distributions of the edge weights.

**Theorem 9.** (Equivalent Definition II of β-Distribution-UMST) Given a connected uncertain graph \( \tilde{G} = (V,E,\xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i \), \( i = 1,2,\ldots,m \), respectively, and a predetermined confidence level \( \beta \in [0,1] \), a spanning tree \( T^0 \) is an uncertain β-distribution minimum spanning tree if and only if

\[
\sum_{e_i \in T^0} \int_{\beta}^{1} \Phi_i^{-1}(\alpha) d\alpha \leq \sum_{e_j \in T} \int_{\beta}^{1} \Phi_j^{-1}(\alpha) d\alpha \tag{35}
\]

holds for any spanning tree \( T \).

**Proof.** It follows immediately from Remark 3 and Theorem 8. \( \square \)

Following from the path optimality condition for classical MST, we extend it to the β-distribution-UMST and get a similar path optimality condition, called the β-distribution-path optimality condition. It is also an equivalent definition of the β-distribution-UMST.

**Theorem 10.** (β-Distribution-Path Optimality Condition) Given a connected uncertain graph \( \tilde{G} = (V,E,\xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i \),
i = 1, 2, ..., m, respectively, and a predetermined confidence level \( \beta \in [0, 1] \), a spanning tree \( T^0 \) is an uncertain \( \beta \)-distribution-minimum spanning tree if and only if

\[
\int_\beta^1 \Phi_0(\alpha)d\alpha - \int_\beta^1 \Phi_j(\alpha)d\alpha \leq 0, \ \forall e_j \in E \setminus T^0, \ e_i \in P_j
\]

where \( E \setminus T^0 \) is the set of non-tree edges, and \( P_j \) is the tree path of non-tree edge \( e_j \).

**Proof.** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \), we can construct a corresponding deterministic graph \( \hat{G} = (V, E, \int_\beta^1 \Phi_0^{-1}(\alpha)d\alpha) \) consisting of the same vertex set \( V \) and edge set \( E \) with uncertain graph \( \tilde{G} \), as well as the deterministic edge weight vector \( \int_\beta^1 \Phi_0^{-1}(\alpha)d\alpha = (\int_\beta^1 \Phi_1^{-1}(\alpha)d\alpha, \int_\beta^1 \Phi_2^{-1}(\alpha)d\alpha, \ldots, \int_\beta^1 \Phi_m^{-1}(\alpha)d\alpha)^T \).

If \( T^0 \) is the \( \beta \)-distribution-UMST of \( \tilde{G} \), according to Theorem 9, \( \sum_{e_i \in T} \int_\beta^1 \Phi_i^{-1}(\alpha)d\alpha \leq \sum_{e_i \in T} \int_\beta^1 \Phi_j^{-1}(\alpha)d\alpha \) holds for any spanning tree \( T \). Then \( T^0 \) is also the MST of \( \hat{G} \). Following from Theorem 2, inequality (36) is obtained.

On the other hand, if \( T^0 \) satisfies (36), according to Theorem 2, \( T^0 \) is the MST of \( \hat{G} \). Then we have \( \sum_{e_i \in T} \int_\beta^1 \Phi_i^{-1}(\alpha)d\alpha \leq \sum_{e_i \in T} \int_\beta^1 \Phi_j^{-1}(\alpha)d\alpha \) holds for any spanning tree \( T \). It follows from Theorem 9 that \( T^0 \) is the \( \beta \)-distribution-UMST of \( \tilde{G} \).

According to Theorem 10, the \( \beta \)-distribution-UMST problem can be transformed to a classical MST problem following from the \( \beta \)-distribution-path optimality condition as follows.

**Theorem 11.** Given a connected uncertain graph \( \tilde{G} = (V, E, \xi) \) where \( \xi_i \) are with regular uncertainty distributions \( \Phi_i \), \( i = 1, 2, \ldots, m \), respectively, and a predetermined confidence level \( \beta \in [0, 1] \), a spanning tree \( T^0 \) is the uncertain \( \beta \)-distribution minimum spanning tree of \( \tilde{G} \) if and only if \( T^0 \) is the minimum spanning tree of a deterministic graph \( \hat{G} = (V, E, \int_\beta^1 \Phi_0^{-1}(\alpha)d\alpha) \), where the deterministic edge weight vector is

\[
\int_\beta^1 \Phi_i^{-1}(\alpha)d\alpha = \left( \int_\beta^1 \Phi_1^{-1}(\alpha)d\alpha, \int_\beta^1 \Phi_2^{-1}(\alpha)d\alpha, \ldots, \int_\beta^1 \Phi_m^{-1}(\alpha)d\alpha \right)^T.
\]

**Proof.** It follows immediately from the proving process of Theorem 10.

Theorem 11 describes the relationship between the \( \beta \)-distribution-UMST of an uncertain graph and its counterpart of a deterministic graph, which also provides an effective method to solve the \( \beta \)-distribution-UMST problem. In terms of this theorem, if we intend to find the \( \beta \)-distribution-UMST of a given uncertain graph \( \tilde{G} = (V, E, \xi) \) with a predetermined confidence level \( \beta \in [0, 1] \), where \( \xi_i \), \( i = 1, 2, \ldots, m \), are uncertain variables with regular uncertainty distributions \( \Phi_i \), respectively, we can turn to find the MST of its corresponding deterministic graph by employing an existing algorithm for the classical MST problem (e.g., the Kruskal algorithm). The solving process is summarized as follows.
Solution to the $\beta$-distribution-UMST problem via deterministic graph transformation

Step 1. Calculate $\int_{0}^{1} \Phi^{-1}_i(\alpha) d\alpha$, $i = 1, 2, \ldots, m$, respectively, and then obtain a deterministic graph $\bar{G} = (V, E, \int_{0}^{1} \Phi^{-1}_i(\alpha) d\alpha)$.

Step 2. Use the Kruskal algorithm to find the minimum spanning tree of $\bar{G}$, denoted by $T^0$.

Step 3. Return $T^0$ as the uncertain $\beta$-distribution-minimum spanning tree of $\tilde{G}$.

6. Numerical Examples

In this section, we give some numerical examples of the uncertain distribution-minimum spanning tree problems to illustrate the conclusions presented above. A connected graph $\tilde{G}$ consisting of 6 vertices and 9 edges is shown in Fig. 5. The edge weights are independent uncertain variables, denoted as $\xi_i, i = 1, 2, \ldots, 9$.

![Uncertain graph for numerical examples](image)

Suppose that $\xi_i, i = 1, 2, \ldots, 9$, are all linear uncertain variables as listed in Table 1. It is known that the expected value of a linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is

$$E[\xi] = (a + b)/2.$$  \hspace{1cm} (38)

In addition, the inverse uncertainty distributions of linear uncertain variables can be calculated on the basis of Eq. (2). As a result, we obtain $\Phi^{-1}_i(\alpha)$ for a given confidence level $\alpha$ as well as $\int_{0}^{1} \Phi^{-1}_i(\alpha) d\alpha$ for a given confidence level $\beta, i = 1, 2, \ldots, 9$. For instance, letting $\alpha = 0.8$ and $\beta = 0.5$, the values of $\Phi^{-1}_i(0.8)$ and $\int_{0.5}^{1} \Phi^{-1}_i(\alpha) d\alpha$ as well as the expected values $E[\xi_i]$ are shown in Table 1.

Example 1. In order to find the overall-distribution-UMST of the graph $\tilde{G}$ in Fig. 5, we can solve the problem via the solution process proposed for the $\beta$-distribution-UMST problem by supposing that $\beta = 0$ since the overall-distribution UMST is a special case of the $\beta$-distribution-UMST according to Remark 2. On the
other hand, we can also find the overall-distribution-UMST by solving the expected UMST problem (see Ref. 28 for details) since the expected UMST is equivalent to the overall-distribution-UMST in terms of Theorem 5. The two ways yield the same result as shown in Fig. 6. There are two spanning trees, \( T_0^1 = \{e_2, e_3, e_7, e_9\} \) and \( T_0^2 = \{e_2, e_4, e_6, e_7, e_9\} \), both of which are the overall-distribution UMST (and also the expected UMST) with the same expected minimum weight

\[
= 2 \times 9 + 8.5 + 2 \times 8 = 42.5.
\]

**Example 2.** Considering the \( \alpha \)-UMST problem, it is clear that the found UMST may be different with respect to different confidence levels \( \alpha \). For example, given a confidence level \( \alpha = 0.8 \), the values of \( \Phi_i^{-1}(0.8) \) are listed in Table 1. Via the approach of deterministic graph transformation (see Ref. 28 for details), we can find

---

Table 1. Parameter values in Fig. 5.

<table>
<thead>
<tr>
<th>Edge i</th>
<th>( \xi_i )</th>
<th>( E[\xi_i] )</th>
<th>( \Phi_i^{-1}(0.8) )</th>
<th>( \int_{0.8}^{1} \Phi_i^{-1}(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>L(10, 12)</td>
<td>11</td>
<td>11.6</td>
<td>5.75</td>
</tr>
<tr>
<td>2</td>
<td>L(7, 11)</td>
<td>9</td>
<td>10.2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>L(4, 14)</td>
<td>9</td>
<td>12</td>
<td>5.75</td>
</tr>
<tr>
<td>4</td>
<td>L(7, 10)</td>
<td>8.5</td>
<td>9.4</td>
<td>4.625</td>
</tr>
<tr>
<td>5</td>
<td>L(9, 13)</td>
<td>11</td>
<td>12.2</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>L(6, 12)</td>
<td>9</td>
<td>10.8</td>
<td>5.25</td>
</tr>
<tr>
<td>7</td>
<td>L(3, 13)</td>
<td>8</td>
<td>11</td>
<td>5.25</td>
</tr>
<tr>
<td>8</td>
<td>L(7, 13)</td>
<td>10</td>
<td>11.8</td>
<td>5.75</td>
</tr>
<tr>
<td>9</td>
<td>L(3, 13)</td>
<td>8</td>
<td>11</td>
<td>5.25</td>
</tr>
</tbody>
</table>
that the 0.8-UMST consists of edges $e_2$, $e_4$, $e_7$, and $e_9$, namely $T_2^0$ in Example 2. However, if the confidence level $\alpha$ changes into 0.9, another spanning tree $T_3^0$, consisting of edges $e_1$, $e_2$, $e_4$, $e_6$, and $e_9$, becomes the new solution. The results of the $\alpha$-UMST problem of the graph $\tilde{G}$ in Fig. 5 with different confidence levels are listed in Table 2, where $W_\alpha(T^0, \xi)$ is the $\alpha$-minimum spanning tree weight.

Figure 7 presents the weight distributions of the spanning trees $T_1^0$, $T_2^0$ and $T_3^0$ in Table 2 (to make the comparison more clearly, the case of $\alpha < 0.5$ is not shown in the figure). It can be seen that if $0.5 \leq \alpha \leq 0.875$, $T_2^0$ is the $\alpha$-UMST, and if $0.875 \leq \alpha \leq 1$, $T_3^0$ is the $\alpha$-UMST. In other words, the $\alpha$-minimum spanning tree weight $W_\alpha(T^0, \xi)$ is determined by different spanning trees with the varying of confidence level $\alpha$. Therefore, it is obvious that the ideal UMST of the graph $\tilde{G}$ in Fig. 5 does not actually exist.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$T^0_1$</th>
<th>$T^0_2$</th>
<th>$T^0_3$</th>
<th>$W_\alpha(T^0_1, \xi)$</th>
<th>$W_\alpha(T^0_2, \xi)$</th>
<th>$W_\alpha(T^0_3, \xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>${e_2, e_3, e_4, e_7, e_9}$</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td>${e_2, e_1, e_2, e_4, e_6, e_9}$</td>
<td>42.5</td>
<td>45.8</td>
<td>55.5</td>
</tr>
<tr>
<td>0.6</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td>${e_2, e_1, e_2, e_4, e_6, e_9}$</td>
<td></td>
<td>49.1</td>
<td>52.4</td>
</tr>
<tr>
<td>0.7</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td></td>
<td></td>
<td>49.1</td>
<td>52.4</td>
</tr>
<tr>
<td>0.8</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td></td>
<td></td>
<td>49.1</td>
<td>52.4</td>
</tr>
<tr>
<td>0.9</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td>${e_2, e_4, e_6, e_7, e_9}$</td>
<td></td>
<td></td>
<td>49.1</td>
<td>52.4</td>
</tr>
</tbody>
</table>

Fig. 7. Weight distributions of the spanning trees $T^0_1$, $T^0_2$, and $T^0_3$. 
**Example 3.** Following from the proposed solution to the $\beta$-distribution-UMST problem, we can find the $\beta$-distribution-UMST of the graph $\tilde{G}$ in Fig. 5 via the deterministic graph transformation method. Taking a given confidence level $\beta = 0.5$ for instance, the values of $\int_{0.5}^{1} \Phi^{-1}_i(\alpha)d\alpha$ and the corresponding deterministic graph with edge vector $\int_{0.5}^{1} \Phi^{-1}_i(\alpha)d\alpha$ are shown in Table 1 and Fig. 8(a), respectively. From Theorem 10 we know that the MST of Fig. 8(a) (as shown in Fig. 8(b)) is also the 0.5-distribution-UMST of Fig. 5. The 0.5-distribution-UMST consists of edges $e_2, e_4, e_6, e_7, \text{and } e_9$ (denoted by $T_2^0$ in Examples 2 and 3) with the 0.5-distribution weight

$$
\sum_{\varepsilon_i \in T_2^0} \int_{0.5}^{1} \Phi^{-1}_i(\alpha)d\alpha = \int_{0.5}^{1} \Phi^{-1}_2(\alpha)d\alpha + \int_{0.5}^{1} \Phi^{-1}_4(\alpha)d\alpha + \int_{0.5}^{1} \Phi^{-1}_6(\alpha)d\alpha \\
+ \int_{0.5}^{1} \Phi^{-1}_7(\alpha)d\alpha + \int_{0.5}^{1} \Phi^{-1}_9(\alpha)d\alpha \\
= 5 + 4.625 + 5.25 + 5.25 = 25.375. 
$$

(40)

As for the spanning tree $T_1^0$, we have

$$
\sum_{\varepsilon_i \in T_1^0} \int_{0.5}^{1} \Phi^{-1}_i(\alpha)d\alpha = \int_{0.5}^{1} \Phi^{-1}_2(\alpha)d\alpha + \int_{0.5}^{1} \Phi^{-1}_3(\alpha)d\alpha + \int_{0.5}^{1} \Phi^{-1}_4(\alpha)d\alpha \\
+ \int_{0.5}^{1} \Phi^{-1}_7(\alpha)d\alpha + \int_{0.5}^{1} \Phi^{-1}_9(\alpha)d\alpha \\
= 5 + 5.75 + 4.625 + 5.25 + 5.25 = 25.875,
$$

(41)

which implies that $T_1^0$ is a worse solution than $T_2^0$ to the $\beta$-distribution-UMST problem. In other words, considering the tail value at risk from the perspective of risk management, $T_2^0$ is more appropriate to be treated as a UMST.
From the results in Examples 2 and 3, we know that both $T_0 = \{e_2, e_3, e_4, e_7, e_9\}$ and $T_2 = \{e_2, e_4, e_6, e_7, e_9\}$ are the expected UMST and 0.5-UMST of the graph $\tilde{G}$ in Fig. 5. That is, according to the decision criteria with respect to expected UMST and $\alpha$-UMST with $\alpha = 0.5$, $T_0$ and $T_2$ have the same performance and thus both of them are treated as the UMST. However, based on the notion of $\beta$-distribution-UMST, it can be found that only $T_2$ is treated as the UMST with a confidence level $\beta = 0.5$, since the 0.5-distribution weight of $T_0$ is less than that of $T_2$.

In summary, by comparing the results in Examples 2, 3 and 4, it can be seen that the proposed notion of distribution-UMST provides an alternative effective way to address the UMST problem.

7. Conclusion
In this paper, we discussed the uncertain minimum spanning tree problem where edge weights are uncertain variables, and proposed three types of uncertain distribution-minimum spanning trees, i.e., max-distribution-UMST, overall-distribution-UMST and $\beta$-distribution-UMST, by referring to the ideal UMST. It was shown that the overall-distribution-UMST is a special case of the $\beta$-distribution-UMST, and also equivalent to the expected UMST. For the general case of the $\beta$-distribution-UMST, some equivalent definitions as well as the path optimality condition are proposed. Based on the path optimality condition, which shows that the notion of the $\beta$-distribution-UMST can be characterized by a set of constraints on non-tree edges and their corresponding tree paths, it was proved that the $\beta$-distribution-UMST problem has an equivalent counterpart in its corresponding deterministic graph, and thus it can be transformed to a classical MST problem and then be solved efficiently by some well-known algorithms.

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