Arithmetic operations for LR mixed fuzzy random variables via mean chance measure with applications

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Abstract. This paper introduces arithmetic operations for LR mixed fuzzy random variables commonly used in practice for modeling fuzzy stochastic phenomena. The operations are proposed based on mean chance measure, which as a natural extension of both the probability of a random event and the credibility of a fuzzy event, measures the mean or expected (in the sense of probability) credibility that the fuzzy random event occurs. Following from the proposed operational laws, the mean chance distributions of LR mixed fuzzy random variables can be deduced explicitly rather than obtained by simulation, which will provide greater convenience for the decision making or optimization in mixed fuzzy and random environments. Furthermore, some important conclusions on the expected value operator defined via the mean chance measure as well as applications of the proposed arithmetic operations in system reliability analysis are also presented.

Keywords: Fuzzy random variable, mean chance, operational law, expected value, reliability analysis

1. Introduction

This paper investigates arithmetic operations for a class of special fuzzy random variables called LR mixed fuzzy random variables. In this section, we first introduce the background and related work with respect to the issue concerned in this paper. Subsequently, the main contributions of this paper are summarized.

1.1. Background and related work

Randomness and fuzziness are two important types of uncertainties encountered in real life, and it is often required to consider both simultaneously. To describe fuzzy stochastic phenomena, fuzzy random variables were introduced by Kwakernaak [8]. Following that, the concept of fuzzy random variable was developed in a rigorous framework by Puri and Ralescu [29]. See Kruse and Meyer [7], and Liu and Liu [25] for more details. So far this concept has been extensively studied in the literature, and has also found many applications such as reliability analysis [23], project scheduling [6], and production planning [9]. For more details on the development of fuzzy random variables, the readers may refer to [4].

While handling decision-making or optimization problems in a hybrid uncertain environment with both randomness and fuzziness, in order to quantify the chance that a fuzzy random event occurs, the concept of chance measure was proposed by Liu [16]. Following that, three types of chance measures, including primitive chance [16], equilibrium chance [26] and mean chance [27], were introduced. The chance measures together with the concept of fuzzy random
The three types of chance measures describe the chance of a fuzzy random event from different aspects. Figure 1 illustrates the geometric interpretation of the three chance measures, where $\text{Ch}^p$, $\text{Ch}^e$ and $\text{Ch}^m$ denote the primitive chance, the equilibrium chance, and the mean chance, respectively. We can see that the primitive chance of the fuzzy random event $\xi \in B$ is a function from $[0, 1]$ to $[0, 1]$ (as the curve shown in Fig. 1), while the equilibrium chance and the mean chance are both scalar values. The value of primitive chance at $\alpha$, i.e., $\text{Ch}^p\{\xi \in B\}(\alpha)$, measures the maximum credibility that a fuzzy random event occurs under a given probability level $\alpha$. The equilibrium chance $\text{Ch}^e\{\xi \in B\}$ measures the maximum credibility as well as the maximum probability that the fuzzy random event occurs at the same time (the point that $\text{Ch}^p\{\xi \in B\}(\alpha) = \alpha$ as shown in Fig. 1). Both of these measures reflect the local property of the event. On the other hand, the mean chance $\text{Ch}^m\{\xi \in B\}$ measures the mean or expected (in the sense of probability) credibility that the fuzzy random event occurs, by which the whole property of the event is concerned. It equals to the area surrounded by two axes and the curve of primitive chance as shown in Fig. 1. It has been shown in Liu and Liu [27] that the mean chance has the advantage over the primitive chance and the equilibrium chance from the point of view of some mathematical properties.

Based on the mean chance theory, Liu [24] presented a new class of two-stage fuzzy random programming models with applications to the capacitated location-allocation problem. Huang et al. [5] employed the mean chance measure to construct a risk model with fuzzy random individual claim amount in the area of insurance. Lan et al. [9] formulated a class of uncertain production planning problems as a fuzzy random production planning model, which maximizes the mean chance of the total costs less than a given allowable investment level. Ojha et al. [28] considered mean chance in some transportation problems with fuzzy-stochastic cost. These works illustrate the great significance of mean chance measure in the area of fuzzy random optimization.

To calculate the chance of a fuzzy random event and compute uncertain functions arising in the area of fuzzy random programming, Liu [16, 17] also constructed a spectrum of fuzzy random simulations. By simulation, we can find critical values (e.g., optimistic value and pessimistic value) as well as the expected value of a fuzzy random variable; we can also check whether a constraint with fuzzy random variables holds with a given confidence level. This idea was also followed and developed by many researchers (see, e.g., [6, 11, 27]). However, due to the difficulties of calculating values and gradients (or subgradients) of chance functions, general mathematical programming methods cannot be applied directly for solving fuzzy random programming problems. Therefore, some hybrid intelligent algorithms were developed to overcome these difficulties (this can also be seen in [6, 11, 27]). However, it is well-known that simulation is a time-consuming process and difficult to use in order to get an exact value.

1.2. Contributions

Motivated by these concerns mentioned above, different from the ideas with regard to fuzzy random simulation, we present in this paper the arithmetic operations for a class of LR mixed fuzzy random variables as well as for their special cases called strictly monotonic LR mixed fuzzy random variables via mean chance measure. The main contributions of this paper are as follows:

Firstly, we define some special types of fuzzy random variables, namely, LR mixed fuzzy random variables and strictly monotonic LR mixed fuzzy random variables, and show that they can be used to model systems containing both random and fuzzy inexactness and have extensive applications in the real world. As mentioned previously, randomness and fuzziness are two important types of uncertainties in real life. Considering a system involving both randomness and fuzziness (see, e.g., [1, 12, 15]), which
can be characterized by random variables and fuzzy variables separately, it is obvious that the performance of the whole system can be described as a typical fuzzy random phenomenon, and further, such phenomenon can be modeled by a mixture of fuzzy variables and random variables. For such systems, we illustrate that the suggested concept of LR mixed fuzzy random variables together with arithmetic operations for them provide an effective way to handling these inexactness.

Secondly, we present operational laws for LR mixed fuzzy random variables, which provide an alternative approach to calculating the chance of a fuzzy random event. Following from the proposed operational laws, the chance distributions of LR mixed fuzzy random variables can be explicitly calculated rather than obtained by simulation. Our work parallels the operational law of uncertain random variables introduced by Liu [22], which made an important contribution to the uncertainty theory founded by Liu [19] in 2007. Inspired by the work of Liu [19] and Liu [22], in this paper we show that analogous operational laws are also valid for LR mixed fuzzy random variables. However, fuzzy random variables concerned in this paper totally differ from uncertain random variables, whether with respect to their theoretical foundations, or with respect to applications. Fuzzy random variables combine the probability theory and the fuzzy set theory, whereas uncertain random variables follow from the probability theory and the uncertainty theory. Theoretically, the fuzzy set theory initiated by Zadeh [35] and the uncertainty theory founded by Liu [19] are developed on the basis of different axioms. The critical difference is that whether the measure of union of events is necessarily the maximum of measures of individual events [20]. For applications, the fuzzy set theory has been well applied in a wide variety of real problems to deal with imperfect knowledge in the real world, whereas the uncertainty theory are mainly applied to modeling human uncertainty in the scenarios where such uncertainties do not behave like fuzziness.

Moreover, we also give some important conclusions on the expected value operator defined via mean chance as well as equivalent ways for calculating the expected values of LR mixed fuzzy random variables.

Lastly, we present applications of the proposed arithmetic operations to reliability analysis for a system containing both random and fuzzy factors. For such a system which is hard to be analyzed by traditional approaches other than simulation, we illustrate that the reliability analysis can be easily handled on the basis of the proposed arithmetic operations.

The rest of this paper is organized as follows. In Section 2, some basic concepts of fuzzy variable and fuzzy random variables are briefly reviewed. In Section 3, a class of LR mixed fuzzy random variables as well as the notion of strictly monotonic LR mixed fuzzy random variables are defined. In Section 4, the operational laws for calculating the mean chance distributions of LR mixed fuzzy random variables are proposed. In Section 5, equivalent forms of the expected value operator as well as theorems for the expected value calculation are suggested. Finally, applications of the proposed arithmetic operations in system reliability analysis are illustrated in Section 6.

2. Preliminaries

In this section, we briefly review the related concepts of fuzzy variable and fuzzy random variable, which will be used throughout this paper.

2.1. Fuzzy variables

Let the triplet $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ be a possibility space, where $\Theta$ is a nonempty set, $\mathcal{P}(\Theta)$ is the power set of $\Theta$, and Pos is a possibility measure [36]. A fuzzy variable is defined as a function from a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to the set of real numbers.

Let $\tau$ be a fuzzy variable with membership function $\mu$, and $B$ a set of real numbers. Then the credibility of a fuzzy event $\{\theta \in \Theta | \tau(\theta) \in B\}$ (denoted as $\{\tau \in B\}$ for simplicity) is defined by Liu and Liu [21] as

$$\text{Cr}\{\tau \in B\} = \frac{1}{2} (\text{Pos}\{\tau \in B\} + \text{Nec}\{\tau \in B\})$$

where Pos and Nec represent the possibility measure [36] and the necessity measure [37], respectively. It has been proved that the credibility measure is increasing and self-dual[21], and

$$\text{Cr}\{\tau \in B\} = \frac{1}{2} \left( \sup_{x \in B} \mu(x) + 1 - \sup_{x \notin B} \mu(x) \right).$$

**Definition 1.** (Liu [17]) The credibility distribution $\Upsilon : \mathfrak{R} \rightarrow [0, 1]$ of a fuzzy variable $\tau$ is defined by

$$\Upsilon(x) = \text{Cr}\{\theta \in \Theta | \tau(\theta) \leq x\}.$$

Here, $\Upsilon(x)$ is the credibility that the fuzzy variable $\tau$ takes a value less than or equal to $x$. It is a nondecreasing function on $\mathfrak{R}$ with $\Upsilon(-\infty) = 0$ and...
\( \Upsilon(+\infty) = 1 \). If \( \Upsilon \) is continuous and strictly increasing on \( \{x \mid 0 < \Upsilon(x) < 1\} \), it is said to be regular [38].

The expected value of a fuzzy variable is defined via the credibility distribution as follows.

**Definition 2.** (Liu and Liu [21]) Let \( \tau \) be a fuzzy variable. Then the expected value of \( \tau \) is defined by

\[
\bar{E}[\tau] = \int_0^{+\infty} \text{Cr}[\tau \geq r]dr - \int_{-\infty}^0 \text{Cr}[\tau \leq r]dr
\]

(1)

provided that at least one of the two integrals is finite.

Note that if only one of the two integrals in (1) is finite, the expected value is infinite. Whereas, if both of them are infinite, the right-hand side of (1) is of form \( -\infty - \infty \), and thus the expected value is not defined.

**Definition 3.** (Liu [18]) The fuzzy variables \( \tau_1, \tau_2, \ldots, \tau_n \) are said to be independent if and only if

\[
\text{Cr}[\tau_i \in B_i, i = 1, 2, \ldots, n] = \min_{1 \leq i \leq n} \text{Cr}[\tau_i \in B_i]
\]

for any sets \( B_1, B_2, \ldots, B_n \) of \( \mathfrak{R} \).

Besides, fuzzy sets and fuzzy numbers have found wide applications in basic and applied research. For example, in topology, fuzzy topological spaces have been studied by Tripathy et al. [32, 33]. In mathematical analysis, sequences of fuzzy numbers have been introduced and their properties have been investigated by Tripathy et al. [30, 31] in recent years. Practical applications of fuzzy variables on the basis of credibility measure can be seen in [10, 13, 14, 39].

2.2. Fuzzy random variables

The concept of fuzzy random variable may be regarded as an extension of both fuzzy variable and random variable. It is a mathematical description for fuzzy stochastic phenomenon, in which fuzziness and randomness appear simultaneously. Roughly speaking, a fuzzy random variable is a random element taking values of fuzzy variable. For our purpose, we use the following definition of fuzzy random variable.

**Definition 4.** (Liu and Liu [25]) Let \( (\Omega, \mathcal{A}, \Pr) \) be a probability space, where \( \Omega \) is a nonempty set, \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets (called events) of \( \Omega \), and \( \Pr \) is a probability measure, and let \( \mathcal{F}_v \) be a collection of fuzzy variables defined on a possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \). A fuzzy random variable is a mapping \( \xi : \Omega \rightarrow \mathcal{F}_v \) such that for any Borel set \( B \) of \( \mathfrak{R} \), \( \text{Cr}[\xi(\omega) \in B] \) is a measurable function of \( \omega \).

Note that for each \( \omega \), the value \( \text{Cr}[\xi(\omega) \in B] \) is the credibility that the fuzzy variable \( \xi(\omega) \) belongs to the Borel set \( B \). According to the definition of fuzzy random variable, we know that \( \text{Cr}[\xi(\omega) \in B] \) is a measurable function of \( \omega \), and hence \( \text{Cr}[\xi(\cdot) \in B] \) is a random variable [25].

**Remark 1.** A random variable is a measurable function from a probability space to the set of real numbers, and, as such, it is a special fuzzy random variable because any real value is a special fuzzy variable. Moreover, if a fuzzy random variable \( \xi \) is a function from a probability space \( (\Omega, \mathcal{A}, \Pr) \) to the singleton \( \{\tau\} \), i.e., \( \xi(\omega) \equiv \tau \), \( \forall \omega \in \Omega \), where \( \tau \) is a fuzzy variable, \( \xi \) degenerates to a fuzzy variable.

For simplicity, the mean chance \( \text{Ch}^m \) of a fuzzy random event is denoted by \( \text{Ch} \), and the term chance refers to mean chance in the remainder of this paper.

**Definition 5.** (Liu and Liu [27]) Let \( \xi \) be a fuzzy random variable, and \( B \) a Borel set of \( \mathfrak{R} \). Then the mean chance of the fuzzy random event \( \{\xi \in B\} \) is defined as

\[
\text{Ch}[\{\xi \in B\}] = \int_0^1 \Pr[\omega \in \Omega | \text{Cr}[\xi(\omega) \in B] \geq r]dr.
\]

Here, \( \{\xi \in B\} \) denotes the fuzzy random event that \( \xi \) takes values in the Borel set \( B \).

**Remark 2.** Note that \( \xi(\omega) \) is a fuzzy variable and \( \text{Cr}[\xi(\omega) \in B] \) is a measurable function of \( \omega \), and thus \( \text{Cr}[\xi(\cdot) \in B] \) is a random variable. The chance \( \text{Ch}[\{\xi \in B\}] \) is in fact the expected value of the random variable \( \text{Cr}[\xi(\cdot) \in B] \), i.e.,

\[
\text{Ch}[\{\xi \in B\}] = \bar{E}[\text{Cr}[\xi(\cdot) \in B]],
\]

where \( \bar{E} \) denotes the expected value operator in the sense of probability.

**Remark 3.** It is easy to see that this definition of mean chance is a natural extension of both the probability of a random event and the credibility of a fuzzy event. If the fuzzy random variable degenerates to a random variable \( \eta \), then \( \text{Ch}[\{\eta \in B\}] = \Pr[\eta \in B] \). If the fuzzy random variable degenerates to a fuzzy variable \( \tau \), then \( \text{Ch}[\{\tau \in B\}] = \text{Cr}[\tau \in B] \).

According to the definition of mean chance measure and the basic properties of probability measure and credibility measure, we have the following conclusions.

**Theorem 1.** (Liu and Liu [27]) Let \( \xi \) be a fuzzy random variable. Then
(1) the chance measure \( \text{Ch}\{\xi \in B\} \) is an increasing set function of Borel set \( B \) and
\[
\text{Ch}\{\xi \in \emptyset\} = 0, \quad \text{Ch}\{\xi \in \mathcal{B}\} = 1;
\]

(2) the chance measure \( \text{Ch}\{\xi \in B\} \) is self-dual. That is, for any Borel set \( B \), we have
\[
\text{Ch}\{\xi \in B\} + \text{Ch}\{\xi \in B^c\} = 1
\]
where \( B^c \) is the complement of \( B \).

Following from the mean chance measure, the chance distribution of a fuzzy random variable is just its probability distribution of a random variable.

Remark 4. As a particular case, the chance distribution of a fuzzy variable is just its credibility distribution, that is, \( \Phi(x) = \text{Cr}\{\xi \leq x\} \).

3. LR mixed fuzzy random variables

In this section, we define a class of LR mixed fuzzy random variables as well as their special cases called the strictly monotonic LR mixed fuzzy random variables, both of which are special fuzzy random variables commonly used in practice. Before introducing these new types of fuzzy random variables, we first discuss some properties of the LR fuzzy numbers.

3.1. LR fuzzy number

As a class of the most commonly used fuzzy numbers in practice, the LR fuzzy numbers initialized by Dubios and Prade [2] can be defined as follows.

Definition 7. (Dubios and Prade [3]) Let \( L \) (and \( R \)) be a shape function from \( \mathcal{B}^+ \) to \([0, 1]\) such that
\[
\begin{align*}
(1) & \quad L(x) \text{ is decreasing;} \\
(2) & \quad L(0) = 1; \\
(3) & \quad L(x) < 1, \quad \forall x > 0; \\
(4) & \quad L(x) > 0, \quad \forall x < 1; \\
(5) & \quad L(1) = 0 \quad \text{or} \quad [L(x) > 0, \forall x \quad \text{and} \quad L(+\infty) = 0].
\end{align*}
\]

(1) A fuzzy number \( \gamma \) is defined as follows.

\[
\text{Ch}\{\gamma \in \mathcal{B}\} = \frac{1}{x} \int_0^x \left( \begin{array}{cc}
L\left(\frac{m-x}{\alpha}\right) & \text{if } x \leq m \\
R\left(\frac{x-m}{\beta}\right) & \text{if } x \geq m
\end{array} \right)
\]

where \( m \) is called the center of \( \gamma \), and \( \alpha \) and \( \beta \) are called the left and right spreads, respectively. Symbolically, \( \gamma \) is denoted by \((m, \alpha, \beta)_{LR}\).

Remark 5. Definition 7 defines two classes of shape functions, one of which describes a bounded support of a fuzzy number, whereas the other is unbounded. The shape function with bounded support is formulated by a decreasing function \( L \) (or \( R \)) from \( \mathcal{B}^+ \) to \([0, 1]\) such that \( L(0) = 1; L(x) < 1 \) for all \( x > 0 \); \( L(x) > 0 \) for all \( x < 1 \); \( L(1) = 0 \). The shape function with unbounded support is formulated by a decreasing function \( L \) (or \( R \)) from \( \mathcal{B}^+ \) to \([0, 1]\) such that \( L(0) = 1; L(x) < 1 \) for all \( x > 0 \); \( L(x) > 0 \) for all \( x < 1 \); \( L(x) > 0 \) for all \( x \) and \( L(+\infty) = 0 \). Examples for these two classes of shape functions are respectively given as follows.

Example 1. A triangular fuzzy number \((a, b, c)\) with membership function \( \mu(x) \) as shown in Fig. 2(a) is with a bounded support. It can be denoted by \((b, a, c-b)_{LR} \), where the shape functions \( L \) and \( R \) are
\[
L(x) = R(x) = \max\{1 - x, 0\}.
\]

Example 2. A Gaussian fuzzy number with membership function \( \mu(x) = e^{-\left(\frac{x-a}{b}\right)^2} \) as shown in Fig. 2(b) is with an unbounded support. It can be denoted by \((a, b, b)_{LR} \), where the shape functions \( L \) and \( R \) are
\[
L(x) = R(x) = e^{-x^2}, \quad x \geq 0.
\]

Definition 8. (Zhou et al. [38]) A fuzzy number \( \gamma \) is defined as a regular LR-type if for \( m, \alpha > 0, \beta > 0 \) in \( \mathcal{B}^+ \), its membership function is
\[
\mu_{\gamma}(x) = \left\{ \begin{array}{ll}
L\left(\frac{m-x}{\alpha}\right) & \text{if } x \leq m \\
R\left(\frac{x-m}{\beta}\right) & \text{if } x \geq m
\end{array} \right.
\]

where \( m \) is the center of \( \gamma \), and \( \alpha \) and \( \beta \) are called the left and right spreads, respectively. Symbolically, \( \gamma \) is denoted by \((m, \alpha, \beta)_{LR}\).

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\[
L(x) = R(x) = \max\{1 - x, 0\}.
\]

Example 2. A Gaussian fuzzy number with membership function \( \mu(x) = e^{-\left(\frac{x-a}{b}\right)^2} \) as shown in Fig. 2(b) is with an unbounded support. It can be denoted by \((a, b, b)_{LR} \), where the shape functions \( L \) and \( R \) are
\[
L(x) = R(x) = e^{-x^2}, \quad x \geq 0.
\]

Definition 8. (Zhou et al. [38]) A fuzzy number \( \gamma \) is defined as a regular LR-type fuzzy number if and only if any one of the following assertions holds,
\[
\begin{align*}
(1) & \quad \text{Its credibility distribution } \gamma(x) = \text{Cr}\{\gamma \leq x\} \quad (x \in \mathcal{B}) \text{ is strictly increasing on } \{x|0 < \gamma(x) < 1\}; \\
(2) & \quad \text{The inverse function } \gamma^{-1}(\alpha), \text{ called the inverse credibility distribution, exists and is strictly increasing on } (0, 1);
\end{align*}
\]
(3) It is an LR fuzzy number and its shape functions $L$ and $R$ are continuous and strictly decreasing on the open intervals $\{x|0 < L(x) < 1\}$ and $\{x|0 < R(x) < 1\}$, respectively.

**Example 3.** The most commonly used triangular fuzzy numbers and Gaussian fuzzy numbers are regular LR fuzzy numbers, and with a strictly increasing credibility distribution on $\{x|0 < \Upsilon(x) < 1\}$. Figure 3 illustrates the credibility distributions of the triangular fuzzy number and the Gaussian fuzzy number shown in Fig. 2. However, the trapezoidal fuzzy number is neither a regular LR fuzzy number nor an LR fuzzy number according to Definitions 7 and 8.

In order to propose the operational law for LR fuzzy numbers, strictly monotone functions are defined as follows by Zhou et al. [38].

**Definition 9.** (Zhou et al. [38]) A real-valued function $f(x_1, x_2, \ldots, x_n)$ is said to be strictly monotone if it is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, that is,

$$f(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n) \leq f(y_1, \ldots, y_m, y_{m+1}, \ldots, y_n)$$

whenever $x_i \leq y_i$ for $i = 1, 2, \ldots, m$ and $x_i \geq y_i$ for $i = m+1, \ldots, n$, and

$$f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) < f(y_1, \ldots, y_m, y_{m+1}, \ldots, y_n)$$

whenever $x_i < y_i$ for $i = 1, 2, \ldots, m$ and $x_i > y_i$ for $i = m+1, \ldots, n$.

For a fuzzy variable formulated as a strictly monotone function of independent regular LR fuzzy numbers, its credibility distribution can be calculated according to the following operational law given by Zhou et al. [38].

**Theorem 3.** (Zhou et al. [38]) Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent regular LR fuzzy numbers with credibility distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_k$ and strictly decreasing with respect to $x_{k+1}, x_{k+2}, \ldots, x_n$, then
Definition 10. \( \tau = f(\tau_1, \tau_2, \ldots, \tau_n) \)

is a regular LR fuzzy number with inverse credibility distribution

\[
\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).
\]

Remark 6. If \( f(x_1, x_2, \ldots, x_n) \) is strictly increasing with respect to \( x_1, x_2, \ldots, x_n \), then the inverse credibility distribution of \( \tau = f(\tau_1, \tau_2, \ldots, \tau_n) \) is equal to \( \Upsilon^{-1}(\alpha) = f(\Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) \). If \( f(x_1, x_2, \ldots, x_n) \) is strictly decreasing with respect to \( x_1, x_2, \ldots, x_n \), then the inverse credibility distribution of \( \tau \) is equal to \( \Upsilon^{-1}(\alpha) = f(\Upsilon_1^{-1}(1-\alpha), \Upsilon_2^{-1}(1-\alpha), \ldots, \Upsilon_n^{-1}(1-\alpha)) \).

3.2. LR mixed fuzzy random variable: Definition and examples

Now, based on the concept of LR fuzzy number, we define some special fuzzy random variables as follows.

Definition 10. (LR mixed fuzzy random variable) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent LR fuzzy variables on a possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \). If \( f \) is a measurable function, then

\[
\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)
\]

is called an LR mixed fuzzy random variable defined by

\[
\xi(\omega) = f(\eta_1(\omega), \eta_2(\omega), \ldots, \eta_m(\omega), \tau_1, \tau_2, \ldots, \tau_n), \quad \forall \omega \in \Omega.
\]

It is clear that an LR mixed fuzzy random variable is a special fuzzy random variable, since it is a measurable function from a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) to the set of fuzzy variables on a possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \).

Example 4. Given a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) where \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_m\} \), if \( \tau_1, \tau_2, \ldots, \tau_n \) are triangular fuzzy numbers, then the function

\[
\xi(\omega) = \begin{cases} 
\tau_1, & \text{if } \omega = \omega_1 \\
\tau_2, & \text{if } \omega = \omega_2 \\
\vdots & \\
\tau_m, & \text{if } \omega = \omega_m 
\end{cases}
\]

is clearly an LR mixed fuzzy random variable.

Example 5. A system consists of two components. It is known that the lifetime of the first one follows an exponential distribution and is denoted by a random variable \( \eta \). However, the second one is a new type developed recently and there is without adequate data to estimate its lifetime statistically. For simplicity, we can formulate it as a triangular fuzzy number \( \tau = (a, b, c) \) by utilizing some expert opinions, where \( a \) and \( c \) are the lower and upper bounds of the lifetime respectively, and \( b \) is the value with most possibility. This is an appropriate and commonly used way in practice. If these two components constitute a parallel system, the lifetime of the whole system \( \eta \lor \tau \) is an LR mixed fuzzy random variable. If they constitute a series system, the lifetime of the whole system becomes \( \eta \land \tau \), which is also an LR mixed fuzzy random variable.

Example 6. Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables, and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent LR fuzzy variables. Then both of the minimum

\[
\xi = \eta_1 \land \eta_2 \land \cdots \land \eta_m \land \tau_1 \land \tau_2 \land \cdots \land \tau_n \quad (2)
\]

and the maximum

\[
\xi = \eta_1 \lor \eta_2 \lor \cdots \lor \eta_m \lor \tau_1 \lor \tau_2 \lor \cdots \lor \tau_n \quad (3)
\]

are LR mixed fuzzy random variables.

Further more, special cases of the LR mixed fuzzy random variables, called strictly monotonic LR mixed fuzzy random variables are defined as follows.

Definition 11. (Strictly monotonic LR mixed fuzzy random variable) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent regular LR fuzzy variables on a possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \). If \( f \) is a measurable function and is strictly monotone with respect to \( \tau_1, \tau_2, \ldots, \tau_n \), then

\[
\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)
\]

is called a strictly monotonic LR mixed fuzzy random variable.

Example 7. Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables, and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent regular LR fuzzy variables. Then the sum

\[
\xi = \eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n
\]

is a strictly monotonic LR mixed fuzzy random variable.

Example 8. Let \( \eta_1, \eta_2, \ldots, \eta_m \) be positive independent random variables, and let \( \tau_1, \tau_2, \ldots, \tau_n \)
be positive independent regular LR fuzzy variables. Then the product
\[ \xi = \eta_1 \eta_2 \cdots \eta_m \tau_1 \tau_2 \cdots \tau_n \]
is a strictly monotonic LR mixed fuzzy random variable.

**Example 9.** A production line consists of three processes, the first two of which are automatically processed by robots, and the processing times can be formulated by two random variables \( \eta_1 \) and \( \eta_2 \) respectively. The last process is quality check and handled manually by workers. Since this process is more complex and may be affected by all kinds of factors (such as environment, worker’s skill and emotion), it is hard to obtain an accurate estimation of the processing time in this stage. Similarly, it can be formulated as a triangular fuzzy number \( \tau = (a, b, c) \) by utilizing some expert opinions, where \( a, b \) and \( c \) are the lower bound, most possible value and upper bound of the processing time, respectively. Then the total processing time of a product in this case can be denoted as \( \eta_1 + \eta_2 + \tau \), and it is a strictly monotonic LR mixed fuzzy random variable.

So far we have defined some special types of fuzzy random variables, which have been shown to have extensive applications. In the following sections, operations for these fuzzy random variables are introduced. Applications to real-world problems will be illustrated as well.

**4. Operational law**

In this section, we propose operational laws for LR mixed fuzzy random variables, which provide novel approaches to analytically calculating the chance distributions of LR mixed fuzzy random variables. Based on the mean chance measure of fuzzy random events, we have the following theorem for LR mixed fuzzy random variables first.

**Theorem 4.** (Operational law for LR mixed fuzzy random variables) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent LR fuzzy variables. Then an LR mixed fuzzy random variable
\[ \xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \]
has a chance distribution
\[
\Phi(x) = \int_{\eta_1, \ldots, \eta_m} \int_{\tau_1, \ldots, \tau_n} F(x; y_1, y_2, \ldots, y_m) \prod_{1}^{m} d\Psi_i(y_i) d\Psi_j(y_j) d\Psi_k(y_k) \tag{4}
\]
where \( F(x; y_1, y_2, \ldots, y_m) \) is the credibility distribution of the fuzzy variable \( f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n) \).

**Proof.** For any given real numbers \( y_1, y_2, \ldots, y_m \), it is clear that the function \( f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) \) is a fuzzy variable. Denote the credibility distribution of \( f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) \) by \( F(x; y_1, y_2, \ldots, y_m) \). Then, according to the definition of mean chance of a fuzzy random event (Definition 5), the chance distribution of \( \xi \) is
\[
\Phi(x) = \text{Ch}\{\xi \leq x\} = E[F(x; \eta_1, \eta_2, \ldots, \eta_m)] .
\]
This is just (4).

Following from the operational law on independent regular LR fuzzy variables, the above theorem can be further deduced to the following one, which provides the operational law for strictly monotonic LR mixed fuzzy random variables.

**Theorem 5.** (Operational law for strictly monotonic LR mixed fuzzy random variables) Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent regular LR fuzzy variables with credibility distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \), respectively. If \( f \) is a measurable function and is strictly monotone with respect to the fuzzy variables, then the strictly monotonic LR mixed fuzzy random variable
\[ \xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \]
has a chance distribution
\[
\Phi(x) = \int_{\eta_1, \ldots, \eta_m} \int_{\tau_1, \ldots, \tau_n} F(x; y_1, y_2, \ldots, y_m) \prod_{1}^{m} d\Psi_i(y_i) d\Psi_j(y_j) d\Psi_k(y_k) \tag{5}
\]
where \( F(x; y_1, y_2, \ldots, y_m) \) is the credibility distribution of the fuzzy variable \( f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n) \) and is determined by its inverse function
\[
F^{-1}(\alpha; y_1, y_2, \ldots, y_m) = f \left( y_1, y_2, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_k^{-1}(\alpha) \right)
\]
\[
\Upsilon_k^{-1}(1 - \alpha), \ldots, \Upsilon_n^{-1}(1 - \alpha) \tag{6}
\]
provided that \( f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n) \) is strictly increasing with respect to \( \tau_1, \tau_2, \ldots, \tau_k \).
and strictly decreasing with respect to $\tau_{k+1}$, $\tau_{k+2}$, \ldots, $\tau_n$.

**Proof.** For any given real numbers $y_1, y_2, \ldots, y_m$, since $f$ is strictly increasing with respect to $\tau_1$, $\tau_2$, \ldots, $\tau_k$ and strictly decreasing with respect to $\tau_{k+1}$, $\tau_{k+2}$, \ldots, $\tau_n$, it follows from the operational law on regular LR fuzzy numbers (Theorem 3) that the inverse credibility distribution of $f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$ can be formulated as (6).

According to Theorem 4, this theorem holds. □

**Remark 7.** If $f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$ is strictly increasing with respect to $\tau_1, \tau_2, \ldots, \tau_n$, then $F^{-1}(\alpha; y_1, y_2, \ldots, y_m)$ in Theorem 5 is equal to

$$f(y_1, y_2, \ldots, y_m, \tau_1^{-1}(\alpha), \ldots, \tau_n^{-1}(\alpha)).$$

If $f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$ is strictly decreasing with respect to $\tau_1, \tau_2, \ldots, \tau_n$, then $F^{-1}(\alpha; y_1, y_2, \ldots, y_m)$ in Theorem 5 is equal to

$$f(y_1, y_2, \ldots, y_m, \tau_1^{-1}(1-\alpha), \ldots, \tau_n^{-1}(1-\alpha)).$$

**Remark 8.** The operational law for strictly monotonic LR mixed fuzzy random variables is an natural extension of both that for independent random variables and that for independent regular LR fuzzy variables. When only randomness is involved in the function $f$, the operational law presented in Theorem 5 degenerates to a special case for independent random variables as follows. If $\xi = f(\eta_1, \eta_2, \ldots, \eta_m)$, and $\eta_1, \eta_2, \ldots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, then $\xi$ is a random variable with probability distribution

$$\Phi(x) = \int_{\mathbb{R}^n} |f(y_1, \ldots, y_m) \leq x| d\Psi_1(y_1) \cdots d\Psi_m(y_m).$$

On the other hand, when only fuzziness is involved in the function $f$, the operational law for strictly monotonic LR mixed fuzzy random variables (Theorem 5) degenerates to the operational law for independent regular LR fuzzy variables proposed by Zhou et al. [38] as presented in Theorem 3.

In the following discussions, we present some examples to illustrate the calculation of chance distributions of LR mixed fuzzy random variables following from the above operational laws.

**Theorem 6.** Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent regular LR fuzzy variables with credibility distributions $\gamma_1, \gamma_2, \ldots, \gamma_n$. Then the minimum defined in Equation (2) is an LR mixed fuzzy random variable whose chance distribution is

$$\Phi(x) = \Psi(x) + \gamma_1(x) - \Psi(x)\gamma_1(x),$$

where $\Psi$ is the probability distribution of $\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m$ determined by

$$\Psi(x) = 1 - (1 - \Psi_1(x))(1 - \Psi_2(x)) \cdots (1 - \Psi_m(x))$$

and $\gamma$ is the credibility distribution of $\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n$ determined by

$$\gamma_1(x) = \gamma_1(x) \vee \gamma_2(x) \vee \cdots \vee \gamma_n(x).$$

**Proof.** Let $\eta = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m$ and $\tau = \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n$. Denote the probability distribution of $\eta$ and the credibility distribution of $\tau$ by $\Psi$ and $\gamma$, respectively. According to Theorem 4, the chance distribution of $\eta \wedge \tau$ can be formulated as

$$\Phi(x) = \int_{-\infty}^{+\infty} F(x; y) d\Psi(y)$$

where $F(x; y)$ is the credibility distribution of the fuzzy variable $y \wedge \tau$, and $y$ is a real number. It is easy to verify that

$$F(x; y) = \begin{cases} \gamma(x), & \text{if } x < y \\ 1, & \text{if } x \geq y. \end{cases}$$

By substituting (11) into (10), (7) is obtained.

Since $\eta_1, \eta_2, \ldots, \eta_m$ are independent random variables, it is clear that (8) holds.

Since $\tau_1, \tau_2, \ldots, \tau_n$ are independent LR fuzzy variables, and $\tau = \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n$ is a strictly increasing function with respect to $\tau_1, \tau_2, \ldots, \tau_n$, the credibility distribution of $\tau$ can be obtained as shown in (9), according to the operational law on independent regular LR fuzzy variables (Theorem 3). □

**Example 10.** Let $\eta$ be a random variable with uniform distribution $U(0, 10)$, and $\tau$ be a triangular fuzzy number $(0, 10, 20)$. The probability distribution of $\eta$ and the credibility distribution of $\tau$ are shown in Figs. 4(a) and (b), respectively.

Then the chance distribution of $\xi = \eta \wedge \tau$ is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.15x - 0.005x^2, & \text{if } 0 \leq x < 10 \\ 1, & \text{if } x \geq 10, \end{cases}$$

which has been depicted in Fig. 5(a).
Theorem 7. Let \( \eta_1, \eta_2, \ldots, \eta_m \) be independent random variables with probability distributions \( \Psi_1, \Psi_2, \ldots, \Psi_m \), and let \( \tau_1, \tau_2, \ldots, \tau_n \) be independent regular LR fuzzy variables with credibility distributions \( \Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n \). Then the maximum defined in Equation (3) is an LR mixed fuzzy random variable whose chance distribution is

\[
\Phi(x) = \Psi(x) \Upsilon(x),
\]

where \( \Psi \) is the probability distribution of \( \eta_1 \lor \eta_2 \lor \cdots \lor \eta_m \) determined by

\[
\Psi(x) = \Psi_1(x) \Psi_2(x) \cdots \Psi_m(x)
\]

and \( \Upsilon \) is the credibility distribution of \( \tau_1 \lor \tau_2 \lor \cdots \lor \tau_n \) determined by

\[
\Upsilon(x) = \Upsilon_1(x) \land \Upsilon_2(x) \land \cdots \land \Upsilon_n(x).
\]

Proof. It can be proved similarly to the proof of Theorem 6.

---

Example 11. Similarly, let \( \eta \) be a random variable with uniform distribution \( U(0, 10) \), and \( \tau \) be a triangular fuzzy number \( (0, 10, 20) \). Then the chance distribution of \( \xi = \eta \lor \tau \) is

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
0.005x^2, & \text{if } 0 \leq x < 10 \\
0.05x, & \text{if } 10 \leq x < 20 \\
1, & \text{if } x \geq 20,
\end{cases}
\]

which has been depicted in Fig. 5(b). Figure 5 compares the chance distributions of the minimum and maximum of \( \eta \) and \( \tau \).

Example 12. Let \( \eta \) be a random variable with probability distribution \( \Psi \), and \( \tau \) be a regular LR fuzzy variable with credibility distribution \( \Upsilon \). Then the sum

\[
\xi = \eta + \tau
\]
is a strictly monotonic LR mixed fuzzy random variable whose chance distribution can be obtained as follows.

According to the operational law for strictly monotonic LR mixed fuzzy random variables, we have

$$
\Phi(x) = \int_{-\infty}^{+\infty} F(x; y) d\Psi(y),
$$

where $F(x; y)$ is the credibility distribution of the fuzzy variable $y + \tau$ and is determined by $F^{-1}(\alpha; y) = y + \Upsilon^{-1}(\alpha)$. Consequently, we have $F(x; y) = \Upsilon(x - y)$, and get the chance distribution of $\xi = \eta + \tau$ as

$$
\Phi(x) = \int_{-\infty}^{+\infty} \Upsilon(x - y) d\Psi(y).
$$

Similarly, assume that $\eta$ has the uniform distribution $U(0, 10)$, and $\tau$ is a triangular fuzzy number $(0, 10, 20)$. Then the sum $\xi = \eta + \tau$ has the following chance distribution

$$
\Phi(x) = 0.1 \int_{x-10}^{x} \Upsilon(y) dy
$$

where $\Upsilon(y)$ is the credibility distribution of $\tau$ (see Fig. 4(b)) with

$$
\Upsilon(y) = \begin{cases} 
0, & \text{if } y < 0 \\
0.05y, & \text{if } 0 \leq y < 20 \\
1, & \text{if } y \geq 20.
\end{cases}
$$

(13)

Substituting (13) into (12), we get

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
0.0025x^2, & \text{if } 0 \leq x < 10 \\
0.05x - 0.25, & \text{if } 10 \leq x < 20 \\
-0.0025x^2 & \text{if } 20 \leq x < 30 \\
+0.15x - 1.25, & \text{if } x \geq 30.
\end{cases}
$$

(13)

The chance distribution of $\xi = \eta + \tau$ is shown in Fig. 6(a).

**Example 13.** Let $\eta$ be a positive random variable with probability distribution $\Psi$, and let $\tau$ be a regular LR fuzzy variable with credibility distribution $\Upsilon$. Then the product

$$
\xi = \eta \tau
$$

is a strictly monotonic LR mixed fuzzy random variable whose chance distribution can be obtained as follows.

According to (5), we have

$$
\Phi(x) = \int_{-\infty}^{+\infty} F(x; y) d\Psi(y),
$$

where $F(x; y)$ is the credibility distribution of the fuzzy variable $\eta \tau$ and is determined by

$$
F^{-1}(\alpha; y) = y \Upsilon^{-1}(\alpha).
$$

Consequently, we have

$$
\Phi(x) = \int_{-\infty}^{+\infty} \Upsilon(x/y) d\Psi(y).
$$

Similarly, if $\eta$ has the uniform distribution $U(0, 10)$, and $\tau$ is a triangular fuzzy number $(0, 10, 20)$, then the chance distribution of $\xi = \eta \tau$ is

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
0.005x & \text{if } 0 \leq x < 200 \\
(1 + \ln 200 - \ln x), & \text{if } x \geq 200.
\end{cases}
$$

The chance distribution of $\xi = \eta \tau$ is shown in Fig. 6(b). Figure 6 compares the chance distributions of the sum and product of $\eta$ and $\tau$.

As shown above, following from the proposed operational laws, the chance distributions of LR mixed fuzzy random variables can be deduced from the probability distributions of the random variables and the credibility distributions of the fuzzy variables.
which characterize the randomness and fuzziness of the LR mixed fuzzy random variables, respectively. This will provide great convenience for decision making or optimization in mixed fuzzy and random environments.

5. Expected value operator

As a fundamental concept for fuzzy random variables, the expected value of a fuzzy random variable has been defined in several ways, for example in Kwakernaak [8], Puri and Ralescu [29], and Kruse and Meyer [7], all of which are defined as fuzzy numbers. To obtain a scalar expected value which is finite, provided that at least one of the above two integrals is finite.

According to the definition of chance distribution (see Definition 6), an equivalent formula of (14) can be immediately obtained as follows,

\[
E[\xi] = \int_{-\infty}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx
\]

(14)

provided that at least one of the above two integrals is finite.

Theorem 8. Let \(\xi\) be a fuzzy random variable. Then its expected value is defined by

\[
E[\xi] = \int_{0}^{+\infty} \text{Ch}[\xi \geq r]dr - \int_{-\infty}^{0} \text{Ch}[\xi \leq r]dr
\]

(14)

The theorem is verified. \(\square\)

Theorem 9. Let \(\eta_1, \eta_2, \ldots, \eta_m\) be independent random variables with probability distributions \(\Psi_1, \Psi_2, \ldots, \Psi_m\), respectively, and let \(\tau_1, \tau_2, \ldots, \tau_n\) be fuzzy variables. Then the fuzzy random variable

\[
\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)
\]

has an expected value

\[
E[\xi] = \int_{\Psi_m(y_m)}^{\Psi_1(y_1)} \cdots d\Psi_m(y_m)
\]

where \(E[f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)]\) is the expected value of the fuzzy variable \(f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)\) for any given real numbers \(y_1, \ldots, y_m\).

Proof. Write the credibility distribution of \(f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)\) as \(F(x; y_1, y_2, \ldots, y_m)\) for any real numbers \(y_1, y_2, \ldots, y_m\). Then according to the definition of expected value of fuzzy variable (see Definition 2), we have

\[
E[f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)]
\]

\[
= \int_{-\infty}^{0} (1 - F(x; y_1, \ldots, y_m))dx
\]

\[
- \int_{-\infty}^{0} F(x; y_1, \ldots, y_m)dx.
\]

On the other hand, the fuzzy random variable \(\xi = f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)\) has a chance distribution \(\Phi(x) = \int_{\Psi_m(y_m)}^{\Psi_1(y_1)} F(x; y_1, \ldots, y_m)d\Psi_1(y_1) \cdots d\Psi_m(y_m)\).

It follows from the equivalent definition of expected value of fuzzy random variable, formulated in (15), that

\[
E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx
\]

\[
= \int_{0}^{+\infty} \left(1 - \int_{\Psi_m(y_m)}^{\Psi_1(y_1)} F(x; y_1, \ldots, y_m) d\Psi_1(y_1) \cdots d\Psi_m(y_m)\right)dx
\]
variable of fuzzy variable (see Definition 2), we have dom variables with probability distributions increasing with respect to $\tau$

Theorem 10. Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent regular LR fuzzy variables with credibility distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. Then the fuzzy random variable

$$\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$$

has an expected value

$$E[\xi] = \int_{\Xi^m} \int_0^1 f \left( y_1, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \ldots, \Upsilon_n^{-1}(1-\alpha) \right)$$

provided that $f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ is strictly increasing with respect to $\tau_1, \ldots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \ldots, \tau_n$.

Proof. For any real numbers $y_1, \ldots, y_m$, denote the credibility distribution of the fuzzy variable $f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)$ by $\Upsilon(x; y_1, \ldots, y_m)$. Since $f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ is strictly increasing with respect to $\tau_1, \ldots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \ldots, \tau_n$, according to the operational law on independent regular LR fuzzy numbers (see Theorem 3) and the definition of expected value of fuzzy variable (see Definition 2), we have

$$\tilde{E} \left[ f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n) \right]$$

Subsequently, it follows from Theorem 9 that (16) holds.

Remark 9. Note that if $f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ is strictly increasing or strictly decreasing with respect to $\tau_1, \ldots, \tau_n$, then the integrand in the formula of expected value $E[\xi]$ in (16) should be replaced by $f(y_1, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha))$.

So far we have presented several equivalent ways for calculating the expected values of some special fuzzy random variables. Particularly, for the strictly monotonic LR mixed fuzzy random variables, we can explicitly calculate their chance distributions following from the operational law presented in Theorem 5, and then get its expected value according to Theorem 8 or via the equivalent formula in (15). On the other hand, we can also calculate its expected value directly by using Theorem 10.

Example 14. Let $\eta$ be a positive random variable with the probability distribution $\Psi$, and let $\tau$ be a positive regular LR fuzzy variable with the credibility distribution $\Upsilon$. Then it follows from Theorem 10 that $\xi = (\eta + \tau)^2$ is a strictly monotonic LR mixed fuzzy random variable with expected value $E[\xi] = \int_{\Xi^m} \int_0^1 (y + \Upsilon^{-1}(\alpha))^2 d\alpha d\Psi(\eta)$.

More specifically, if $\eta$ has a uniform distribution $U(0, 10)$, and $\tau$ is a triangular fuzzy number (0, 10, 20), then the expected value of $\xi = (\eta + \tau)^2$ is

$$E[\xi] = 0.1 \int_0^{10} \int_0^1 (y + 20\alpha)^2 d\alpha d\Psi = \frac{800}{3}.$$
6. Applications to system reliability analysis

In this section, we illustrate the proposed operations by their applications in system reliability analysis. Consider a series system consisting of 4 components, whose structure is shown in Fig. 7. To improve the reliability of this system, redundancy for the components are provided. That is, for each component, redundant elements (n in total) are added as backup, one of which begins to work only when the active element fails. The elements in a same component are of the same type, and thus they have the same properties concerning their lifetimes. Moreover, the lifetimes of all the elements in the system are assumed to be independent.

As for the lifetimes, on the one hand, it is known that the lifetimes of elements in components 1 and 2 are random variables $\mathcal{N}(50, 25)$ and $\mathcal{N}(55, 36)$, respectively, where $\mathcal{N}(\mu, \sigma^2)$ represents a normal random variable whose probability density function is

$$
\Psi(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left( -\frac{(x - \mu)^2}{2\sigma^2} \right).
$$

On the other hand, since the elements in components 3 and 4 are new types developed recently, there is a lack of adequate data to estimate their lifetimes statistically. Instead, expert opinions are used to provide the estimates, which are formulated as fuzzy variables. It is assumed that the lifetimes of elements in components 3 and 4 are triangular fuzzy variables (35, 50, 65) and (42, 52, 62), respectively.

Denote by $\eta_{1i}$, $\eta_{2i}$, $\tau_{1i}$, and $\tau_{2i}$ the lifetimes of the $i$th elements in components 1, 2, 3, and 4, respectively, $i = 1, 2, \ldots, n$. Then the lifetime of the entire system $T(n)$ can be formulated by

$$
T(n) = \sum_{i=1}^{n} \eta_{1i} \land \sum_{i=1}^{n} \eta_{2i} \land \sum_{i=1}^{n} \tau_{1i} \land \sum_{i=1}^{n} \tau_{2i}.
$$

Now, we consider the chance distribution of the system lifetime (denoted by $\Phi_T$). It follows from the operational law for LR mixed fuzzy random variables (or directly from Theorem 6) that

$$
\Phi_T(x) = \Psi(x) + \Upsilon(x) - \Psi(x) \Upsilon(x),
$$

where $\Upsilon$ is the credibility distribution of $\sum_{i=1}^{n} \tau_{1i} \land \sum_{i=1}^{n} \tau_{2i}$ and

$$
\Upsilon(x) = \begin{cases} 
0, & \text{if } x < 35n \\
(x - 35n)/30n, & \text{if } 35n \leq x < 56n \\
(x - 42n)/20n, & \text{if } 56n \leq x < 62n \\
1, & \text{if } x \geq 62n.
\end{cases}
$$

Here $\Psi$ is the probability distribution of $\sum_{i=1}^{n} \eta_{1i} \land \sum_{i=1}^{n} \eta_{2i}$ and

$$
\Psi(x) = 1 - \left( 1 - \Phi_{N}\left( \frac{x - 50n}{5\sqrt{n}} \right) \right) - \left( 1 - \Phi_{N}\left( \frac{x - 55n}{6\sqrt{n}} \right) \right)
$$

where $\Phi_{N}$ denotes the standard normal distribution.

Figure 8 shows the chance distributions of the system lifetime $T(n)$ for $n = 1, 2, \ldots, 6$. Based on the chance distributions, the expected values of system lifetime with different levels of redundancy can be calculated as shown in Table 1.

To evaluate the reliability of the system, we may further refer to the chance that the system fails before a targeted lifetime $T^o$, i.e., $\text{Ch}\{T(n) < T^o\} = \Phi_T(T^o)$, which is also known as the risk index in the sense of risk analysis; or the confidence of achieving this targeted goal, i.e., $\text{Ch}\{T(n) \geq T^o\} = 1 - \Phi_T(T^o)$. The risk and the confidence with respect to a targeted lifetime $T^o = 200$ are shown in Table 1. It is shown that the risk of system failing before time 200 equals 1 if the level of redundancy $n = 1, 2, 3$, which implies that the system lifetime would certainly be less than 200; however, if $n = 6$, the risk approaches 0. It is clear that if we want to design a system such that the system lifetime exceeds 200 with...
Fig. 8. The chance distributions of the system lifetime $T(n)$.

Table 1
System reliability with different levels of redundancy

<table>
<thead>
<tr>
<th>Level of redundancy: $n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected value of the lifetime: $E[T(n)]$</td>
<td>45.2695</td>
<td>91.5915</td>
<td>137.9285</td>
<td>184.2560</td>
<td>230.5705</td>
<td>276.8730</td>
</tr>
<tr>
<td>Risk of system failing before time 200: $\text{Ch}[T(n) &lt; 200]$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.7619</td>
<td>0.1667</td>
<td>0.0000</td>
</tr>
<tr>
<td>Confidence of lifetime exceeding 200: $\text{Ch}[T(n) \geq 200]$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.2381</td>
<td>0.8333</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

confidence $0.8$, at least 5 elements should be added in each component.

Obviously, the reliability of such a system with different types of uncertainties is hard to be analyzed by traditional approaches other than simulation. For a system involving fuzzy random parameters, a fuzzy random simulation technique, which was suggested by Liu [17], has become the primary approach to reliability analysis (see, e.g., [23, 27, 34]). However, as illustrated in the above examples, such an analysis can be easily handled following from the proposed arithmetic operations for LR mixed fuzzy random variables, which provide an alternative approach to reliability analysis. With these arithmetic operations, the chance distribution of system lifetime as well as the risk of system failing can be explicitly deduced.

7. Conclusion

In this paper we focused on the arithmetic operations for LR mixed fuzzy random variables as well as for their special cases, namely, strictly monotonic LR mixed fuzzy random variables. These arithmetic operations were developed on the basis of the mean chance measure of fuzzy random events, since its advantage relies on mathematical properties and availability in practice. Operational laws, through which the chance distributions of LR mixed fuzzy random variables can be explicitly calculated rather than obtained by simulation, as well as some important conclusions on the expected value operator and equivalent ways for calculating the expected values of LR mixed fuzzy random variables were proposed and illustrated by many examples in this paper. We also showed that the analysis of a system containing both random and fuzzy factors can benefit greatly from the proposed arithmetic operations by specific applications.

Furthermore, we believe that the operational laws as well as other conclusions presented in this paper are of great significance for the fuzzy random decision making or optimization, since fuzzy stochastic phenomena appear frequently in the real world, and most of them can be modeled by a mixture of fuzzy variables and random variables.

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References
