Arithmetic Operations on Triangular Fuzzy Numbers via Credibility Measures: An Inverse Distribution Approach

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Abstract. With extensive applications of fuzzy numbers, many methods for fuzzy arithmetics especially the basic operations have been developed based on Zadeh’s extension principle. Among these methods, the interval arithmetic approach and the standard approximation method are the most important and commonly used exact and approximate methods, respectively. In this paper, regarding the continuous and strictly monotone functions of triangular fuzzy numbers, we propose an inverse distribution approach to deriving the exact or well approximate membership functions for arithmetic results by embedding the credibility measure of fuzzy sets into fuzzy arithmetics. Besides, some non-complicated and complicated examples are given to illustrate the performance of the new approach, together with a detailed comparison with the interval arithmetic approach and the standard approximate method. Furthermore, the inverse distribution approach is also applied to the fuzzy system reliability calculation based on fault tree compared with serval current related researches.

Keywords: Triangular fuzzy number, fuzzy arithmetic, membership function, credibility measure, inverse distribution, reliability analysis

1. Introduction

In 1965, Zadeh [32] proposed the concept of fuzzy sets, and since then it has greatly changed the way of ambiguity and imprecision conventionally considered. Notably, the fuzzy arithmetic, as a significant element of the fuzzy set theory, becomes a well-formalized criterion that allows users to manage uncertain environments more reasonably, which has been widely applied in practice under fuzzy environments such as medical diagnosis, group decision making, transportation problems etc. (see, e.g., [1,3,5,31]).

Initially, the arithmetic operations of fuzzy sets were defined based on Zadeh’s extension principle as a well-established and powerful tool.

Definition 1 (Zadeh [33]) Let \( f : \mathbb{R} \times \mathbb{R} \) be a binary operation over real numbers. Then it can be extended to the operation of fuzzy quantities over the set \( \mathbb{R} \). If we denote for \( A, B \in \mathbb{R} \) the quantity \( C = f(A, B) \), then the membership function \( \mu_C \) is derived from the membership functions \( \mu_A \) and \( \mu_B \) by

\[
\mu_C(z) = \max[\min(\mu_A(x), \mu_B(y)) : x, y \in \mathbb{R}, z = f(x, y)], \quad \forall z \in \mathbb{R}.
\]

According to Definition 1, the operations of \( \max \) and \( \min \) are included in the fuzzy arithmetics of fuzzy numbers which make the arithmetics be non-linear, computationally expensive and hard to be applied. Out of the need for solving practical problems, many approaches have been developed towards fuzzy arithmetics gradually aiming at simplifying the computational process, which can be mainly divided into t-
wo branches, i.e., approximate and exact approaches. Regarding the commonly used $L\tau - R$ fuzzy numbers, Dubois and Prade [12] initially proposed the classic standard approximation method. This method is easy to proceed and reduces computational complexity greatly, and thus is accepted as adequate and widely used in real applications so far [8,28]. Nevertheless too frequent a use of this formula may lead to wrong results according to Dubois and Prade [12]. In order to reduce the error generated from the standard approximation, many other approximate approaches were proposed from different considerations to improve the approximate effect as much as possible. Thinking that the main error source of standard approximation is from approximating polynomial curves with straight lines, Giachetti and Young [14] came up with a parametric representation of fuzzy numbers and provided a method for performing fuzzy arithmetics. Based on the $L^{-1}.R^{-1}$ inverse function arithmetic principle, Chou [9] proposed a canonical representation of multiplication operation on triangular fuzzy numbers (TFN-s). Besides, Guerra and Stefanini [15] presented another method to approximate and represent fuzzy numbers by means of piecewise monotonic interpolations and derived a procedure to control the absolute error associated to the arithmetic operations on fuzzy numbers. In order to obtain the approximations on the real line simply and flexibly, Coroianu et al. [11] discussed piecewise linear 1-knot fuzzy numbers, which have some desirable properties of original fuzzy numbers. Thereafter, Ban et al. [2] further studied the extend weighted $L\tau - R$ approximation of fuzzy numbers computed by the approach on the basis of results in Hilbert spaces, which can be applied to any $L\tau - R$ approximation.

However, no matter which approximate approach is applied, errors away from exact arithmetic results always exist, and sometimes they may be extremely large to produce erroneous results. Thus many exact approaches for fuzzy arithmetics were proposed, among which the most classical one is the interval approach initiated by Kaufmann and Gupta [17]. Instead of outputting the membership functions of arithmetic results, the interval arithmetic approach ends with the exact $b$-cuts of the actual results of basic fuzzy arithmetics, i.e., $\{a, b, c, d\}$ on triangular fuzzy numbers or trapezoidal fuzzy numbers. Following them, lots of theoretical expansions and practical applications of interval arithmetic approach have been developed widely (see, e.g., [4,6,19,25]). Besides, on the basis of credibility measure founded by Liu and Liu [22], Chutia et al. [10] proposed an alternative method to deduce the exact membership functions of results of basic fuzzy arithmetics on one or two TFNs, whose applications are restricted in the arithmetic operators and the number of TFNs. Recently, Zhou et al. [35] provided an operational law for fuzzy arithmetics to analytically and exactly calculate the inverse credibility distribution of some specific arithmetical operations based on the credibility measure, and Wang et al. [29] proposed the operations based on the mean chance measure to calculate the expected value or credibility that a fuzzy random event occurs. Compared with approximate manners, studies on exact manners for fuzzy arithmetics are limited, and do not provide a considerable improvement in the aspect of complexity.

Considering that a relatively exact membership function with uncomplicated calculation process could make an important impact on subsequent work, especially in the area of fuzzy optimization and decision making, in this paper, motivated by the recent results on inverse credibility distribution in Zhou et al. [35], we propose a novel inverse distribution approach for continuous and strictly monotone functions of TFNs, which could deduce the exact or a well approximate membership function of the arithmetic result based on the principle whether the inverse function can be easily derived or not. The underlying fundamentals of this approach mainly lie in the relationships of the membership function, the credibility distribution, and the inverse credibility distribution of a TFN, as well as the operational law presented in [35]. It is also proved that the new approach is applicable to not only TFNs but also any fuzzy numbers with continuous and strictly increasing credibility distributions. Some numerical experiments together with an example of fuzzy reliability analysis are further presented to show that a considerable improvement could be achieved when comparing with the classical standard approximation method, interval arithmetic approach and some other existing methods.

The rest of the paper is organized as follows. Section 2 briefly introduces the usual calculation methods for basic arithmetics on TFNs as well as some relevant concepts. Section 3 describes the details of the inverse distribution approach. Some numerical examples are then provided in Section 4 to illustrate the performance of the proposed approach. Following that, the novel method is further utilized to analyze fuzzy system reliability in Section 5. Finally, conclusions and contributions are listed in Section 6.
2. Preliminaries

In this section, a brief introduction is given for the usual calculation methods for fuzzy arithmetics on TFNs, including the interval arithmetic approach and the standard approximation method. Besides, the credibility measure developed from the possibility measure [33] and necessity measure [34], and the definitions of credibility distributions and inverse credibility distributions of fuzzy numbers are also reviewed, which would be employed in the following sections.

2.1. Two approaches for fuzzy arithmetics on TFNs

A TFN is usually denoted as \( \xi = (a, b, c) \), where \( b \) indicates the center value, and \( a, c \) indicate the lower and upper limit values, respectively. The membership function of \( \xi \) is expressed as

\[
\mu_\xi(x) = \begin{cases} 
\frac{x - a}{b - a}, & \text{if } a \leq x < b \\
\frac{c - x}{c - b}, & \text{if } b \leq x \leq c \\
0, & \text{otherwise.}
\end{cases}
\]

Suppose that \( \xi = (a_1, b_1, c_1) \), \( \eta = (a_2, b_2, c_2) \) are two TFNs. Then according to the standard approximation method, which was initially proposed by Dubois and Prade [12], the results of basic arithmetic operations on \( \xi \) and \( \eta \) can be obtained (as shown in Table 1). Here, it should be noted that with respect to operations \( \oplus \) and \( \oslash \), we only list the results for cases of \( \xi > 0 \) and \( \eta > 0 \), which implies \( a_1 > 0 \) and \( a_2 > 0 \).

![Fig. 1. The h-cuts of \( \xi = (a, b, c) \)](image)

### Table 1

Results of basic arithmetics via the standard approximation method

<table>
<thead>
<tr>
<th>Operations</th>
<th>Arithmetic results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi \oplus \eta )</td>
<td>((a_1 + a_2, b_1 + b_2, c_1 + c_2))</td>
</tr>
<tr>
<td>( \xi \oslash \eta )</td>
<td>((a_1 \times a_2, b_1 \times b_2, c_1 \times c_2))</td>
</tr>
<tr>
<td>( \xi \ominus \eta )</td>
<td>((a_1 - c_2, b_1 - b_2, c_1 - a_2))</td>
</tr>
<tr>
<td>( \xi \oslash \eta )</td>
<td>((a_1, b_1, c_1) \div \frac{b_1}{a_2})</td>
</tr>
</tbody>
</table>

**Example 1:** Let \( \xi = (2, 5, 6) \) and \( \eta = (1, 2, 3) \). According to the standard approximation method, the basic fuzzy arithmetics on them can be obtained as follows,

\[
\begin{align*}
\xi \oplus \eta &= (2 + 1, 5 + 2, 6 + 3) = (3, 7, 9), \\
\xi \oslash \eta &= (2 - 3, 5 - 2, 6 - 1) = (-1, 3, 5), \\
\xi \ominus \eta &= (2 \times 1, 5 \times 2, 6 \times 3) = (2, 10, 18), \\
\xi \oslash \eta &= (2 \div 3, 5 \div 2, 6 \div 1) = \left(\frac{2}{3}, \frac{5}{2}, 6\right).
\end{align*}
\]

The results of arithmetic operations \( \oplus \) and \( \oslash \) of TFNs obtained via the standard approximation method are actually exact, while the results of \( \ominus \) and \( \oslash \) of TFNs are approximate and may be far from the exact values sometimes, which could lead to wrong conclusions or guidances for users.

Besides, it is known that the \( h \)-cuts of a TFN \( \xi = (a, b, c) \) define a set of closed intervals. For any \( h \in [0, 1] \), the interval is \([((b - a)h + a, (b - c)h + c]) \), which is depicted in Figure 1, where the left and right limit values of \( h \)-cuts are denoted as \( \xi^{L}_h \) and \( \xi^{R}_h \), respectively. Then from the perspective of \( h \)-cuts, the interval arithmetic approach was proposed by Kaufmann and Gupta [17] and becomes another commonly used method for fuzzy arithmetics in addition to the standard approximation method. The \( h \)-cuts of results of basic arithmetic operations on two TFNs can be obtained through the interval arithmetic approach and expressed in Table 2.

**Example 2:** Let \( \xi = (2, 5, 6) \) and \( \eta = (1, 2, 3) \). It can be known that the \( h \)-cuts of \( \xi \) and \( \eta \) are \([2 + 3h, 6 - h]\) and \([1 + h, 3 - h]\), respectively. According to the interval arithmetic approach, the \( h \)-cuts of the results of basic fuzzy arithmetics on them can be derived as.
follows,
\[
\xi \oplus \eta = \left[ \frac{\xi^L + \eta^L}{\eta^L}, \frac{\xi^R + \eta^R}{\eta^R} \right] = [3 + 4h, 9 - 2h],
\]
\[
\xi \ominus \eta = \left[ \frac{\xi^L - \eta^L}{\eta^L}, \frac{\xi^R - \eta^R}{\eta^R} \right] = [4h - 1, 5 - 2h],
\]
\[
\xi \odot \eta = \left[ \min \left\{ \frac{\xi^L \eta^L, \xi^L \eta^R, \xi^R \eta^L, \xi^R \eta^R} \right\}, \max \left\{ \frac{\xi^L \eta^L, \xi^L \eta^R, \xi^R \eta^L, \xi^R \eta^R} \right\} \right] = [2 + 5h + 3h^2, 18 - 9h + h^2],
\]
\[
\xi \otimes \eta = \left[ \min \left\{ \frac{\xi^L \eta^L, \xi^L \eta^R, \xi^R \eta^L, \xi^R \eta^R} \right\}, \max \left\{ \frac{\xi^L \eta^L, \xi^L \eta^R, \xi^R \eta^L, \xi^R \eta^R} \right\} \right] = \left[ \frac{2 + 3h}{3 - h}, \frac{6 - h}{1 + h} \right].
\]

Regarding the interval arithmetic approach, the membership functions of arithmetic results are not provided, which may cause inconvenience if users need to utilize the membership functions in their subsequent work. Additionally, the operations of max and min are still included in the results via the interval arithmetic approach, which would make the arithmetic complicated, especially when the signs of the left and right limit values of the fuzzy numbers are not the same or the number of TFNs involved in the arithmetic is substantial.

2.2. Credibility distribution and inverse credibility distribution

Credibility measure was suggested by Liu and Li-\[20\]u \[22\] to measure the credibility that a fuzzy event will occur, defined as the average of possibility measure and necessity measure, i.e., \( \text{Cr}(A) = \frac{1}{2} (\text{Pos}(A) + \text{Nec}(A)) \) holds for any fuzzy event \( A \). Based on the conception of credibility measure, Liu \[20\] further presented the credibility distribution of a fuzzy number, which substantially plays the role of probability distribution in probability theory.

**Definition 2** (Liu \[20\]) The credibility distribution \( \Phi : (-\infty, +\infty) \rightarrow [0, 1] \) of a fuzzy number \( \xi \) is defined by
\[
\Phi(x) = \text{Cr}(\theta \in \Theta | \xi(\theta) \leq x).
\]
That is, \( \Phi(x) \) is the credibility that the fuzzy number \( \xi \) takes a value less than or equal to \( x \). Generally speaking, the credibility distribution of any fuzzy number is non-decreasing. Moreover, regarding fuzzy numbers with continuous and strictly increasing credibility distributions, the definition of inverse credibility distribution was proposed by Zhou et al. \[35\] as follows.

**Definition 3** (Zhou et al. \[35\]) Let \( \xi \) be a fuzzy number with a continuous and strictly increasing credibility distribution \( \Phi \) with respect to \( x \) at which \( 0 < \Phi(x) < 1 \). Then the inverse function \( \Phi^{-1} \) is called the inverse credibility distribution of \( \xi \).

Note that the inverse credibility distribution \( \Phi^{-1} \) is well defined on the open interval \((0, 1)\). If required, we may extend the domain to \([0, 1]\) by letting
\[
\Phi^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi^{-1}(\alpha).
\]

Besides, assume that \( \xi = f(\xi_1, \xi_2, \cdots, \xi_n) \), where \( \xi_i \) are fuzzy numbers with inverse credibility distributions \( \Phi_{\xi_i}^{-1}(\alpha), i = 1, 2, \cdots, n \), respectively. If \( f \) is a continuous and strictly monotone function, then the inverse credibility distribution of \( \xi \), denoted as \( \Phi_{\xi}^{-1} \), can be deduced based on the operational law proved by Zhou et al. \[35\] as follows.

**Theorem 1** (Zhou et al. \[35\]) Let \( \xi_1, \xi_2, \cdots, \xi_n \) be independent fuzzy numbers with continuous and strictly increasing credibility distributions \( \Phi_{\xi_1}, \Phi_{\xi_2}, \cdots, \Phi_{\xi_n}, \) respectively. If the function \( f(x_1, x_2, \cdots, x_n) \) is continuous and strictly increasing with respect to \( x_1, \cdots, x_m \) and strictly decreasing with respect to \( x_{m+1}, \cdots, x_n \), then \( \xi = f(\xi_1, \xi_2, \cdots, \xi_n) \) is a fuzzy number with inverse credibility distribution
\[
\Phi_{\xi}^{-1}(\alpha) = \Phi_{\xi_1}^{-1}(\alpha), \cdots, \Phi_{\xi_{m}}^{-1}(\alpha), \Phi_{\xi_{m+1}}^{-1}(1-\alpha), \cdots, \Phi_{\xi_{n}}^{-1}(1-\alpha),
\]
(3)
where \( \Phi_{\xi_1}^{-1}, \Phi_{\xi_2}^{-1}, \cdots, \Phi_{\xi_n}^{-1} \) are the inverse credibility distributions of \( \xi_1, \xi_2, \cdots, \xi_n \), respectively.
3. An Inverse Distribution Approach

Assume that \( \xi_1, \xi_2, \ldots, \xi_n \) are TFNs, and \( f(x_1, x_2, \ldots, x_n) \) is a continuous and strictly monotone function with respect to \( x_1, x_2, \ldots, x_n \). In order to derive the membership function of fuzzy number \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) analytically from membership functions of \( \xi_1, \xi_2, \ldots, \xi_n \), this section first proves some theorems on relationships of the membership function \( \mu \), credibility distribution \( \Phi \), and inverse credibility distribution \( \Phi^{-1} \) for a special type of fuzzy numbers, and then utilizes the theorems as well as the operational law proposed in [35] to develop a so-called inverse distribution approach. The detailed procedures of this approach are described step by step by taking TFNs for instance, and an elaborated flowchart is finally given as a summary.

3.1. Relations of \( \mu \), \( \Phi \), and \( \Phi^{-1} \)

**Theorem 2** Let \( \xi \) be a fuzzy number with membership function \( \mu \). Then its credibility distribution is

\[
\Phi(x) = \frac{1}{2} \left( \sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y) \right), \quad \forall x \in \mathbb{R}. \tag{4}
\]

**Proof:** Suppose that \( \xi \) is defined on the possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \). It follows from the definitions of possibility measure and necessity measure that for any \( x \in \mathbb{R} \), the possibility of fuzzy event \( \xi \leq x \) is

\[
\text{Pos}\{\xi \leq x\} = \text{Pos}\{\theta \in \Theta \mid \xi(\theta) \leq x\} = \sup_{y \leq x} \mu(y),
\]

and the necessity of fuzzy event \( \xi \leq x \) is

\[
\text{Nec}\{\xi \leq x\} = 1 - \text{Pos}\{\xi > x\} = 1 - \sup_{y > x} \mu(y).
\]

Hence, in the light of the definition of credibility measure (see Liu and Liu [22]) and credibility distribution of a fuzzy number (see Definition 2), it is easy to deduce that

\[
\Phi(x) = C(\xi \leq x)
\]

\[
= \frac{1}{2} \left( \text{Pos}\{\xi \leq x\} + \text{Nec}\{\xi \leq x\} \right)
\]

\[
= \frac{1}{2} \left( \sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y) \right).
\]

**Theorem 3** Let \( \xi \) be a fuzzy number with a continuous and strictly increasing credibility distribution \( \Phi \). Then its membership function is

\[
\mu(x) = \begin{cases} 2\Phi(x), & \text{if } \Phi(x) < 0.5 \\ 2 - 2\Phi(x), & \text{if } \Phi(x) \geq 0.5. \end{cases} \tag{5}
\]

**Proof:** Supposing that \( \xi \) is defined on the possibility space \( (\Theta, \mathcal{P}(\Theta), \text{Pos}) \), we know that

\[
\text{Pos}\{\theta \in \Theta \mid \xi(\theta) = x, x \in \mathbb{R}\} = \sup_{x \in \mathbb{R}} \mu(x) = 1,
\]

which follows that \( \sup_{y \leq x} \mu(y) \lor \sup_{y > x} \mu(y) = 1 \). In other words,

\[
\sup_{y \leq x} \mu(y) = 1 \lor \sup_{y > x} \mu(y) = 1
\]

holds for any \( x \in \mathbb{R} \). If \( \Phi(x) < 0.5 \), according to Eq. (4), it is easy to obtain \( \sup_{y > x} \mu(y) = 1 \). Then we have

\[
\Phi(x) = \frac{1}{2} \left( \sup_{y \leq x} \mu(y) + 1 - 1 \right) = \frac{1}{2} \sup_{y \leq x} \mu(y). \tag{6}
\]

Furthermore, since the credibility distribution of \( \Phi(x) \) is continuous and strictly increasing, the membership function \( \mu(x) \) of \( \xi \) is also continuous and strictly increasing in \( \Phi^{-1}(0), \Phi^{-1}(0.5) \), and \( \sup_{y \leq x} \mu(y) = \mu(x) \). Combining with Eq. (6), we finally get \( \Phi(x) = \frac{1}{2} \mu(x) \), that is \( \mu(x) = 2\Phi(x) \) for \( \Phi(x) < 0.5 \).

Similarly, if \( \Phi(x) \geq 0.5 \), it can be deduced that

\[
\Phi(x) = \frac{1}{2} \left( \sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y) \right)
\]

\[
= \frac{1}{2} \left( 1 + 1 - \sup_{y > x} \mu(y) \right)
\]

\[
= 1 - \frac{1}{2} \sup_{y > x} \mu(y)
\]

\[
= 1 - \frac{1}{2} \mu(x).
\]

Subsequently, we obtain \( \mu(x) = 2 - 2\Phi(x) \).

From Theorems 2 and 3, it can be observed that there is a one-to-one relationship between the membership function \( \mu \) and the credibility distribution \( \Phi \) for a fuzzy number \( \xi \) with a continuous and strictly monotone credibility distribution. That is, provided that one of the two functions is known, the other one can be obtained immediately via Eq. (4) or Eq. (5).
Theorem 4 Let $\xi$ be a fuzzy number. If the inverse credibility distribution of $\xi$ exists, denoted as $\Psi(\alpha)$, which is continuous and strictly increasing with respect to $\alpha$ in $[0, 1]$, then its credibility distribution is

$$
\Phi(x) = \begin{cases} 
0, & \text{if } x < \Psi(0) \\
\Psi^{-1}(x), & \text{if } \Psi(0) \leq x \leq \Psi(1) \\
1, & \text{if } x > \Psi(1),
\end{cases}
$$

(7)

which is continuous and strictly increasing in $[\Psi(0), \Psi(1)]$.

Proof: It follows immediately from the definition of inverse credibility distribution of a fuzzy number (see Definition 3).

According to Definition 3 and Theorem 4, if $\xi$ is a fuzzy number with a continuous and strictly monotone distribution, there also exists a one-to-one relation between its credibility distribution $\Phi$ and the inverse credibility distribution $\Psi$ (i.e., $\Phi^{-1}$), both of which are continuous and strictly increasing functions. Given one of the two functions, the other one is then derived by using the inverse operation directly.

Moreover, combining with the conclusions of Theorems 2 and 3, it follows immediately that for any fuzzy number $\xi$ with a continuous and strictly monotone distribution, the membership function $\mu$, credibility distribution $\Phi$ and inverse credibility distribution $\Phi^{-1}$ may be deduced from any one of the other two functions. The mutual relations of $\mu$, $\Phi$ and $\Phi^{-1}$ will be utilized to design the inverse distribution approach in the following section.

Theorem 5 Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent fuzzy numbers with continuous and strictly increasing credibility distributions $\Phi_{\xi_1}, \Phi_{\xi_2}, \ldots, \Phi_{\xi_n}$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is continuous and strictly monotone with respect to $x_1, x_2, \ldots, x_n$, then the inverse credibility distribution of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ exists and is continuous and strictly increasing in $[0, 1]$.

Proof: Without loss of generality, let $\xi = f(\xi_1, \xi_2)$, and assume that $f(x_1, x_2)$ is strictly increasing and decreasing with respect to $x_1$ and $x_2$, respectively. According to Eq. (3), we have $\Phi_{\xi}^{-1}(\alpha) = f(\Phi_{\xi_1}^{-1}(\alpha), \Phi_{\xi_2}^{-1}(1 - \alpha))$.

Firstly, based on the definition of inverse credibility distribution (see Definition 3) and the strict monotonicity of credibility distribution, it can be known that the inverse credibility distributions $\Phi_{\xi_1}^{-1}(\alpha)$ and $\Phi_{\xi_2}^{-1}(\alpha)$ are both strictly increasing, and so for all $\alpha_1, \alpha_2$ in $[0, 1]$, if $\alpha_1 > \alpha_2$, we have $\Phi_{\xi_1}^{-1}(\alpha_1) > \Phi_{\xi_1}^{-1}(\alpha_2)$ and $\Phi_{\xi_2}^{-1}(1 - \alpha_2) < \Phi_{\xi_2}^{-1}(1 - \alpha_1)$. Besides, on the basis of the strict monotonicity of $f$ we get $f(\Phi_{\xi_1}^{-1}(\alpha_1), \Phi_{\xi_2}^{-1}(1 - \alpha_1)) > f(\Phi_{\xi_1}^{-1}(\alpha_2), \Phi_{\xi_2}^{-1}(1 - \alpha_2))$. That is,

$$
\Phi_{\xi_1}^{-1}(\alpha_1) > \Phi_{\xi_1}^{-1}(\alpha_2).
$$

The continuity of $\Phi_{\xi}^{-1}(\alpha)$ follows immediately from the continuity of $f$, $\Phi_{\xi_1}^{-1}(\alpha)$ and $\Phi_{\xi_2}^{-1}(\alpha)$.

According to Theorems 1 and 5, for a fuzzy number $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$, if the conditions in Theorem 1 (or Theorem 5) hold, then both of the inverse credibility distribution and the credibility distribution of $\xi$ exist, and may be calculated via Eq. (3) and Eq. (7), respectively, in accordance with the inverse credibility distributions of $\xi_1, \xi_2, \ldots, \xi_n$.

Finally, taking advantage of all the theorems proved in this section along with the operational law in Theorem 1, we develop an approach to deriving the membership function of a fuzzy number $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ in the light of membership functions $\mu_{\xi_1}, \mu_{\xi_2}, \ldots, \mu_{\xi_n}$ of fuzzy numbers $\xi_1, \xi_2, \ldots, \xi_n$, whose basic idea is depicted briefly in Figure 2.

As shown in Figure 2, the credibility distributions $\Phi_1, \Phi_\xi$ and the inverse credibility distribution $\Phi_{\xi}^{-1}$ contained in the dashed box are just utilized as intermediaries when deriving $\mu_\xi$ from $\mu_{\xi_1}$, owing to
3.2. Procedures of inverse distribution approach

In Zhou et al. [35], it has been proved that a fuzzy number has a continuous and strictly increasing credibility distribution if and only if it is an L-R fuzzy number and its shape functions L and R are continuous and strictly decreasing on the open intervals \( \{x|0 < L(x) < 1\}\) and \(\{x|0 < R(x) < 1\}\), respectively. Several types of fuzzy numbers possess these properties, a typical one among which is TFNs. Since the arithmetic operations of TFNs are widely discussed in most literature due to its extensive applications while expressing the ambiguous valuations in practical problems, this section takes TFNs as an example to demonstrate the detailed procedures of the proposed approach depicted in Figure 2 step by step as follows.

**Step 1:** Derive the credibility distribution \(\Phi_{\xi_i}\) from \(\mu_{\xi_i}\) for each fuzzy number \(\xi_i\) according to Eq. (4) in Theorem 2.

Without loss of generality, denote the triangular fuzzy numbers considered by \(\xi_i = (a_i, b_i, c_i)\), \(i = 1, 2, \ldots, n\), where \(a_i, b_i, c_i\) represent the lower limit value, the center value, and the upper limit value of \(\xi_i\), respectively. Then according to Eq. (4) and the membership function of TFNs in Eq. (1), for each \(i(1 \leq i \leq n)\), the credibility distribution of \(\xi_i\) is

\[
\Phi_{\xi_i}(x) = \begin{cases} 
0, & \text{if } x < a_i \\
\frac{x - a_i}{2(b_i - a_i)}, & \text{if } a_i \leq x < b_i \\
\frac{x + c_i - 2b_i}{2(c_i - b_i)}, & \text{if } b_i \leq x < c_i \\
1, & \text{if } x \geq c_i,
\end{cases}
\]

which is depicted in Figure 3.

**Step 2:** Derive the inverse credibility distribution \(\Phi_{\xi_i}^{-1}\) for each fuzzy number \(\xi_i\) via the inverse operation on account of the definition of inverse credibility distribution in Definition 3.

From Figure 3, it can be observed that the credibility distribution of \(\xi_i = (a_i, b_i, c_i)\) is continuous and strictly increasing in \(\{x|0 < \Phi(x) < 1\}\) = \((a_i, c_i)\).

Then according to Definition 3, for each \(i(1 \leq i \leq n)\), the inverse credibility distribution of \(\xi_i\) can be deduced through the inverse operation directly as follows,

\[
\Phi_{\xi_i}^{-1}(\alpha) = \begin{cases} 
2(b_i - a_i)\alpha + a_i, & \text{if } 0 \leq \alpha < 0.5 \\
2(c_i - b_i)\alpha + 2b_i - c_i, & \text{if } 0.5 \leq \alpha \leq 1,
\end{cases}
\]

which is depicted in Figure 4.

**Step 3:** Derive the inverse credibility distribution \(\Phi_{\xi}^{-1}\) for \(\xi = f(\xi_1, \xi_2, \ldots, \xi_n)\) according to Eq. (3) in Theorem 1.

The third step of this approach is to derive the inverse credibility distribution of \(\xi = f(\xi_1, \xi_2, \ldots, \xi_n)\), denoted as \(\Phi_{\xi}^{-1}\), where \(f\) is a continuous and strictly monotone function. For convenience, assume that \(f\) is strictly increasing with respect to \(x_1, x_2, \ldots, x_m\), and strictly decreasing with respect to \(x_{m+1}, x_{m+2}\),
\[ \Phi^{-1}_\xi(x) = f(\Phi^{-1}_{\xi_1}(\alpha), \ldots, \Phi^{-1}_{\xi_n}(1-\alpha)), \]
in which \( \Phi^{-1}_{\xi_i}(\alpha), i = 1, 2, \ldots, m, \) and \( \Phi^{-1}_{\xi_i}(1-\alpha), i = m+1, m+2, \ldots, n, \) have been obtained in Step 2 via Eq. (9). Obviously, the inverse credibility distribution \( \Phi^{-1}_\xi(\alpha) \) calculated above is a continuous and strictly increasing function with respect to \( \alpha \) in [0, 1].

**Step 4: Derive the credibility distribution of \( \xi \) via the inverse operation according to Eq. (7) in Theorem 4.**

Going through Step 3, the inverse credibility distribution \( \Phi^{-1}_\xi(x) \) is obtained, which is continuous and strictly increasing based upon Theorem 5. Thus it follows from Theorem 4 that the credibility distribution \( \Phi_\xi(x) \), i.e., the inverse function of \( \Phi^{-1}_\xi(\alpha) \), exists, and may be deduced from the inverse credibility distribution \( \Phi^{-1}_\xi(\alpha) \) directly by using Eq. (7), which is also a continuous and strictly increasing function.

In most cases, the inverse credibility distribution \( \Phi^{-1}_\xi(x) \) is a simple function with respect to \( \alpha \), and thus it is not difficult to deduce \( \Phi_\xi(x) \) via the inverse operation, e.g.,

\[ \Phi^{-1}_\xi(\alpha) = \alpha^2 \Rightarrow \Phi_\xi(x) = \sqrt{x}. \]

As a result, the exact credibility distribution of \( \xi \) is obtained. However, sometimes the function \( \Phi^{-1}_\xi(\alpha) \) would be complicated, e.g.,

\[ \Phi^{-1}_\xi(\alpha) = \alpha^7 - 2\alpha^3. \]

In this case, it may be tough to deduce the precise credibility distribution \( \Phi_\xi(x) \) of \( \xi \) through the inverse operation on \( \Phi^{-1}_\xi(\alpha) \). Regarding this kind of complicated cases, a natural idea is to utilize the techniques of curve fitting. As is well known, many developed software packages may report a well approximate polynomial function by minimizing the sum of squared errors for \( \Phi_\xi(x) \), among which Excel and Matlab are what researchers frequently adopt. Since the 'polyfit' function in Matlab could output a better fit in a least-squares sense (the R-square of fitting results could approach to 1.0 in most cases) compared with Excel, we employ Matlab to get a well approximate expression (instead of an exact one) for the credibility distribution \( \Phi_\xi(x) \) that could be utilized in the following steps. A detailed demonstration for this kind of situation is given in Example 6 of Section 4.

**Step 5: Derive the membership function \( \mu_\xi(x) \) of \( \xi \) according to Eq. (5) in Theorem 3.**

As mentioned in Step 4, the credibility distribution \( \Phi_\xi(x) \) of the fuzzy number \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) exists and is continuous and strictly monotone, and subsequently the exact credibility distribution of \( \xi \) (or a well approximation credibility distribution) is obtained via the inverse operation (or with the aid of software packages). Thus according to Theorem 3, we can straightforwardly get its membership function \( \mu_\xi(x) \) from \( \Phi_\xi(x) \) accordingly via Eq. (5).

### 3.3. Flowchart of the inverse distribution approach

So far, the detailed implementation procedures of the inverse distribution approach to deriving the membership function \( \mu_\xi(x) \) of \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) have been presented step by step, where \( \xi_1, \xi_2, \ldots, \xi_n \) are TFNs and \( f \) is a continuous and strictly monotone function. As a summarization of this section, a flowchart is provided in Figure 5 to show the main scheme of this approach by taking TFNs for instance.

At the end of this section, three critical points related to the proposed method should be emphasized. The first one is the applicable type of fuzzy numbers involved in the function \( f \). Even though the above implementation procedures in Section 3.2 and the flowchart of the approach in Section 3.3 are presented by assuming that \( \xi_1, \xi_2, \ldots, \xi_n \) are TFNs and \( f \) is a continuous and strictly monotone function. As a summarization of this section, a flowchart is provided in Figure 5 to show the main scheme of this approach by taking TFNs for instance.
Input the membership functions

\[ \xi_1, \xi_2, \ldots, \xi_n \]

do complicated cases

\[
\mu_{\xi_1}(x), \mu_{\xi_2}(x), \ldots, \mu_{\xi_n}(x)
\]
of \( \xi_1, \xi_2, \ldots, \xi_n \), and

\( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \)

Output the membership function

\[ \mu_{\xi}(x) \]
of \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \)

Fig. 5. Flowchart of the inverse distribution approach

all the steps except Step 4 are obviously exact function transformation without any approximation. As for Step 4, if the inverse credibility distribution \( \Phi_{\xi}^{-1} \) is a non-complicated function, the credibility distribution \( \Phi_{\xi} \) can be deduced straightforwardly by implementing the inverse operation. In this case, a precise membership function of the arithmetic result will be obtained. Otherwise, the techniques of curve fitting could be utilized to derive a well approximate function for \( \Phi_{\xi} \) with the aid of software packages for convenience. Under this circumstance, the precision of the result is essentially decided by the software adopted. In the present paper, Matlab is recommended on account of its effectiveness and high precision in the function approximation, which will be illustrated by some numerical examples in the following sections.

Additionally, as to the monotone function \( f \), from the perspective of optimization problems, it could be found that almost all objective and constraint function-
s in practical problems are indeed monotone with respect to the fuzzy parameters. Hence our method is extremely practical when dealing with the fuzzy arithmetics in most fuzzy optimization problems in practice.

4. Numerical Examples

In this section, four numerical examples are listed to illustrate the performance and effectiveness of the inverse distribution approach we proposed for fuzzy arithmetics on TFNs, the most commonly used fuzzy numbers due to its intuition, convenience of use and computational simplicity. The first three examples are performed for general non-complicated cases taken from Guerra and Stefanini [15], in which the exact membership functions of arithmetic results can be derived. Properly speaking, numerical examples in the literature that studied fuzzy arithmetics are almost in similar forms with the examples in this paper. Besides, the last example is given to show how to proceed the proposed approach against complicated cases with the help of software. Regarding each example, contrasts and verifications are presented for the results derived from our approach with those derived from the interval arithmetic approach and the standard approximation method, respectively.

Example 3: Let \( \xi_1 = (2, 3, 4) \), and \( f(x) = x^6 \).
Calculate the membership function of \( \xi = f(\xi_1) = (2, 3, 4)^6 \).

Firstly, the credibility distribution of \( \xi_1 \) is deduced through Eq. (8) as

\[
\Phi_{\xi_1}(x) = \begin{cases} 
0, & \text{if } x < 2 \\
\frac{x - 2}{2}, & \text{if } 2 \leq x < 4 \\
1, & \text{if } x \geq 4.
\end{cases}
\]

Secondly, the inverse credibility distribution of \( \xi_1 \) is deduced by implementing the inverse operation on \( \Phi_{\xi_1}(x) \) (or via Eq. (9) directly) as

\[
\Phi_{\xi_1}^{-1}(\alpha) = 2\alpha + 2, \quad 0 \leq \alpha \leq 1.
\]

Thirdly, since \( f \) is continuous and strictly increasing with respect to \( x \) for \( x \geq 0 \), according to Eq. (3), the inverse credibility distribution of \( \xi \) is

\[
\Phi_{\xi}^{-1}(\alpha) = f(\Phi_{\xi_1}^{-1}(\alpha)) = (2\alpha + 2)^6.
\]
Fourthly, through the inverse operation, the credibility distribution of $\xi$ can be conducted immediately as

$$
\Phi_\xi(x) = \begin{cases} 
0, & \text{if } x < 2^6 \\
\frac{\sqrt{x} - 2}{2}, & \text{if } 2^6 \leq x \leq 4^6 \\
1, & \text{if } x > 4^6.
\end{cases}
$$

Finally, based on Eq. (5), the exact membership function of $\xi$ can be derived as

$$
\mu_\xi(x) = \begin{cases} 
\frac{\sqrt{x} - 2}{2}, & \text{if } 2^6 \leq x < 3^6 \\
4 - \frac{\sqrt{x}}{2}, & \text{if } 3^6 \leq x \leq 4^6 \\
0, & \text{otherwise},
\end{cases}
$$

and the center value, the lower and upper limit values of $\xi$ are $3^6$, $2^6$, and $4^6$, respectively.

Next, in order to verify the correctness and effectiveness of the membership function we derive, the respective comparisons with the interval arithmetic approach and the standard approximation method are made and shown in Figure 6.

![Fig. 6. The membership function of $\xi$ in Example 3](image)

In Figure 6, the dot-dashed line depicts the membership function of $\xi$ obtained through the standard approximation method described in Table 1, the solid line presents the membership function of $\xi$ deduced through the novel inverse distribution approach proposed in this paper, while the $h$-cuts of $\xi$ derived based on the interval arithmetic approach described in Table 2 for $h = 0, 0.05, \ldots, 1$ are shown with two points marked with ‘•’. It can be noticed that all the points that indicate the left and right limit values of $h$-cuts of $\xi$ just fall on the solid line, which verifies the correctness of the membership function we deduced. Subsequently, when $2^6 \leq \xi \leq 3^6$, we can see that the membership degrees of $\xi$ deduced from the standard approximation method are lower than the exact values, while when $3^6 \leq \xi \leq 4^6$, the membership degrees of $\xi$ are much higher than the exact values deduced from either our approach or the interval arithmetic approach.

**Example 4:** Let $\xi_1 = (0.4, 0.7, 1.2)$, and $f(x) = x^{-4}$. Calculate the membership function of $\xi = f(\xi_1) = (0.4, 0.7, 1.2)^{-4}$.

Firstly, since $f$ is continuous and strictly decreasing with respect to $x$ for $x \geq 0$, based on Eq. (9), we can get the inverse credibility distribution of $\xi$ as

$$
\Phi_\xi^{-1}(\alpha) = f(\Phi_{\xi_1}^{-1}(1 - \alpha))
$$

and the center value, the lower and upper limit values of $\xi$ are $0.7^{-4}$, $1.2^{-4}$, and $0.4^{-4}$, respectively.

Secondly, through the inverse operation, the credibility distribution of $\xi$ can be conducted immediately as

$$
\Phi_\xi(x) = \begin{cases} 
0, & \text{if } x < 1.2^{-4} \\
1.2 - \frac{1}{\sqrt{x}}, & \text{if } 1.2^{-4} \leq x < 0.7^{-4} \\
3 - \frac{5}{3\sqrt{x}}, & \text{if } 0.7^{-4} \leq x \leq 0.4^{-4} \\
1, & \text{if } x > 0.4^{-4}.
\end{cases}
$$

Finally, based on Eq. (5), the membership function of $\xi$ can be derived as

$$
\mu_\xi(x) = \begin{cases} 
2.4 - \frac{2}{\sqrt{x}}, & \text{if } 1.2^{-4} \leq x < 0.7^{-4} \\
10 - \frac{4}{3\sqrt{x}}, & \text{if } 0.7^{-4} \leq x \leq 0.4^{-4} \\
0, & \text{otherwise},
\end{cases}
$$

and the center value, the lower and upper limit values of $\xi$ are $0.7^{-4}$, $1.2^{-4}$, and $0.4^{-4}$, respectively.

Similarly, a comparative analysis with the interval arithmetic approach and the standard approximation method is made and depicted in Figure 7. The $h$-cuts of $\xi$ deduced based on the interval arithmetic approach are all consistent with that calculated with our approach. Besides, when $1.2^{-4} \leq \xi \leq 0.7^{-4}$, the mem-
bership degrees of $\xi$ deduced from the standard approximation method are lower than the exact values, while when $0.7^{-4} \leq \xi \leq 0.4^{-4}$, the membership degrees of $\xi$ are much higher than the exact values deduced from either our approach or the interval arithmetic approach.

![Fig. 7. The membership function of $\xi$ in Example 4](image)

**Example 5:** Let $\xi_1 = (1,4,9)$, $\xi_2 = (1,2,3)$, and $f(x_1,x_2) = \frac{x_1}{x_2}$. Calculate the membership function of $\xi = f(\xi_1,\xi_2) = \frac{(1,4,9)}{(1,2,3)}$.

In this example, since $f(x_1,x_2)$ is continuous and strictly increasing with respect to $x_1$, and strictly decreasing with respect to $x_2 (x_2 \neq 0)$, based on Eq. (3), the inverse credibility distribution of $\xi$ can be deduced. Then via the inverse operation, the credibility distribution of $\xi$ is conducted immediately as

$$\Phi_\xi(x) = \begin{cases} 
0, & \text{if } x < \frac{1}{3} \\
\frac{3}{2} - \frac{5}{x + 3}, & \text{if } \frac{1}{3} \leq x < 2 \\
\frac{3}{2} - \frac{7}{x + 5}, & \text{if } 2 \leq x \leq 9 \\
1, & \text{if } x > 9.
\end{cases}$$

In the light of Eq. (5), the membership function of $\xi$ can be derived as

$$\mu_\xi(x) = \begin{cases} 
3 - \frac{10}{x + 3}, & \text{if } \frac{1}{3} \leq x < 2 \\
\frac{14}{x + 5} - 1, & \text{if } 2 \leq x \leq 9 \\
0, & \text{otherwise},
\end{cases}$$

which is depicted in Figure 8, and the center value, the lower and upper limit values of $\xi$ are 2, 1/3, and 9, respectively. Similarly, comparisons with the interval arithmetic approach and the standard approximation method are presented, which testify the correctness and effectiveness of the inverse distribution approach once again.

![Fig. 8. The membership function of $\xi$ in Example 5](image)

Until now, regarding three different kinds of strictly monotone functions, i.e., strictly increasing, strictly decreasing, and strictly monotone, three corresponding simple examples are listed to illustrate the process and performance of the proposed inverse distribution approach. Besides, as mentioned previously, sometimes the inverse credibility distribution of $\xi = f(\xi_1,\xi_2,\cdots,\xi_n)$ obtained through Step 3 may be complicated so that the derivation of credibility distribution of $\xi$ is too tough to proceed through the inverse operation. In order to handle this kind of cases, in this paper, we recommend to utilize some well developed software, e.g., Matlab, to obtain a well approximate function for the credibility distribution of $\xi$, and then continue the rest steps of inverse distribution approach to deduce a well approximate membership function for $\xi$. In the following, we present an example to illustrate the process.

**Example 6:** Let $\xi_1 = (2,3,4), \xi_2 = (2,3,5), \xi_3 = (1,4,9), \xi_4 = (1,2,3)$, and $f(x_1,x_2,x_3,x_4) = x_1^2 x_2^2 - x_3 x_4^2$. Calculate the membership function of $\xi = f(\xi_1,\xi_2,\xi_3,\xi_4) = (2,3,4)^6 \times (2,3,5)^2 - (1,4,9) \times (1,2,3)^2$. 

$$\Phi_\xi(x) = \begin{cases} 
0, & \text{if } x < \frac{1}{3} \\
\frac{3}{2} - \frac{5}{x + 3}, & \text{if } \frac{1}{3} \leq x < 2 \\
\frac{3}{2} - \frac{7}{x + 5}, & \text{if } 2 \leq x \leq 9 \\
1, & \text{if } x > 9.
\end{cases}$$

In the light of Eq. (5), the membership function of $\xi$ can be derived as

$$\mu_\xi(x) = \begin{cases} 
3 - \frac{10}{x + 3}, & \text{if } \frac{1}{3} \leq x < 2 \\
\frac{14}{x + 5} - 1, & \text{if } 2 \leq x \leq 9 \\
0, & \text{otherwise},
\end{cases}$$

which is depicted in Figure 8, and the center value, the lower and upper limit values of $\xi$ are 2, 1/3, and 9, respectively. Similarly, comparisons with the interval arithmetic approach and the standard approximation method are presented, which testify the correctness and effectiveness of the inverse distribution approach once again.

![Fig. 8. The membership function of $\xi$ in Example 5](image)
Firstly, going through Steps 1 to 3, we can get the inverse credibility distribution of $\xi$ as

$$\Phi^{-1}_\xi(\alpha) = f(\Phi^{-1}_\xi(\alpha), \Phi^{-1}_\xi(\alpha), \Phi^{-1}_\xi(1 - \alpha),$$

$$\Phi^{-1}_\xi(1 - \alpha))$$

$$= \begin{cases} 
(2\alpha + 2)^5(2\alpha + 2)^2 - (9 - 10\alpha) \\
(3 - 2\alpha)^2, \quad \text{if } 0 \leq \alpha < 0.5 \\
(2\alpha + 2)^5(4\alpha + 1)^2 - (7 - 6\alpha) \\
(3 - 2\alpha)^2, \quad \text{if } 0.5 \leq \alpha \leq 1.
\end{cases}$$

Secondly, utilizing the ‘polyfit’ function of Matlab, the credibility distribution of $\xi$ can be approximately expressed as

$$\Phi_\xi(x) \approx \begin{cases} 
0, \quad \text{if } x < 47 \\
-0.0016x + 0.0243x^{6/7} - 0.1278x^{5/7} + 0.2544x^{4/7} - 0.3873x^{3/7} + 0.2836, \quad \text{if } 47 \leq x < 2171 \\
-0.0002x + 0.0044x^{6/7} - 0.0518x^{5/7} + 0.3440x^{4/7} - 1.3827x^{3/7} + 3.3766x^{2/7} - 4.2662x^{1/7} + 2.1340, \quad \text{if } 2171 \leq x \leq 25599 \\
1, \quad \text{if } x > 25599.
\end{cases}$$

It should be noted that the $R^2$-square of the fitting result above is 1.0, which implies that the precision of approximation is extremely high.

Based on Eq. (5), the membership function of $\xi$ can be derived as

$$\mu_\xi(x) \approx \begin{cases} 
-0.0032x + 0.0486x^{6/7} - 0.2556x^{5/7} + 0.5088x^{4/7} - 0.7746x^{3/7} + 0.5672, \quad \text{if } 47 \leq x < 2171 \\
0.0004x - 0.0088x^{6/7} + 0.1036x^{5/7} - 0.6880x^{4/7} + 2.7654x^{3/7} - 6.7532x^{2/7} + 8.5324x^{1/7} + 6.2680, \quad \text{if } 2171 \leq x \leq 25599 \\
0, \quad \text{otherwise.}
\end{cases}$$

It is easy to deduce that the center value, the lower and upper limit values of $\xi$ are 2171, 47, and 25599, respectively. Similarly, Figure 9 depicts the above approximate membership function of $\xi$ together with the left and right limit values of some exact $h$-cuts derived from the interval arithmetic approach and the approximate membership function derived from the standard approximation method. From Figure 9, it can be seen that for complicated cases, even though the exact membership function cannot be obtained, we can acquire the one with a relatively high accuracy on account of the relation of the solid line and ‘*’ shown in Figure 9 as the most important evidential basis, which provides a strong support to the utilize of our method in the subsequent applications.

Based on the presentations of above four examples including non-complicated cases, i.e., Examples 3~5 and complicated cases, i.e., Example 6, it is obvious that the correctness of results obtained by using the inverse distribution approach gets a great improvement when comparing with the commonly used standard approximation method for both cases. In Section 1, we have mentioned that in the standard approximation methods, the exact membership function curves are roughly approximated with straight lines, which can be observed directly from Figures 6~9. Even though the standard approximation method is simple and easy to proceed, however, at the expense of correctness, so this method is not recommended, especially in some practical engineering programming.

On the other hand, all the arithmetic results via our proposed approach are almost completely coincident with the results obtained through the interval arithmetic approach, including the well approximate membership function. In terms of the interval arithmetic approach, exact expressions for the left and right limit
values of \( h \)-cuts of arithmetic result can be obtained, which may be used to conduct the exact membership function through the inverse function operation in essence. Similarly, for the complicated cases, the inverse function operation may be also tough to proceed, and then some techniques of curve fitting could be adopted through all kinds of software packages. But to our knowledge, this work has not been completed yet in the existing literature. In addition, the interval arithmetic approach merely provides the solutions for the basic arithmetics on TFNs, while for other types of fuzzy arithmetics, there is not a clear instruction. Besides, the operations of \( \max \) and \( \min \) are still included in the interval arithmetic approach, which may cause the calculation complicated, especially when the number of TFNs increases substantially. However, with the aid of the inverse distribution approach proposed in this paper, provided that the function of TFNs is continuous and strictly monotone, the exact or a well approximate membership function (rather than the \( h \)-cuts obtained by the interval arithmetic approach) would be derived through a relatively simple procedure, which could be used in the subsequent applications.

5. Fuzzy System Reliability Analysis

In system reliability analysis problems, owing to the infrequency of the hazard event and the instability of a system, it is always difficult to obtain exact probability of an event which results in the probability varying within a certain range. Thus many researchers address these problems in a fuzzy way (see, e.g., [18,24,27,30]). In this section, an example of fuzzy system reliability introduced by Singer [26] is used to demonstrate the efficiency and accuracy of the inverse distribution method by comparing with the results in [7,26] and those by the interval arithmetic approach.

A scenario of system reliability related to grinding machine safety is illustrated in Singer’s example, whose process can be described as a fault tree in Figure 10, where the basic events \( A \sim H \) contributing to the accident are described specifically in Table 3. According to Chen [7] and Singer [26], the possibilities of occurrence of the basic events are assumed as TFNs and denoted as \( \xi_A \sim \xi_H \), respectively. Since the spread values of fuzzy parameters in [7,26] are set too small, which is not in accordance with practice due to the lack of effective and sufficient historical data in an uncertain environment, this paper enlarges the original spreads of \( \xi_A \sim \xi_H \) to a comparatively reasonable range, see Figure 10.

Let \( X \) indicate the main event that people get hurt when coming to the working area of two grinding machines, whose possibility of occurrence as the system reliability is denoted by \( \xi \). Based upon the fault tree in Figure 10, the truth function of the main event \( X \) can be written as follows,

\[
U = F + G + H, \quad V = C + D, \\
Z = E \times U \times V, \quad X = A + B + Z.
\]

Accordingly, the system reliability \( \xi \) is a function of possibilities of occurrence of basic events calculated as

\[
\xi = f(\xi_A, \xi_B, \xi_C, \xi_D, \xi_E, \xi_F, \xi_G, \xi_H) \\
= 1 - (1 - \xi_A)(1 - \xi_B)(1 - \xi_E(1 - (1 - \xi_F)) \\
(1 - \xi_G)(1 - \xi_H))(1 - (1 - \xi_C)(1 - \xi_D))).
\]

Apparently, the function \( f \) is continuous and strictly increasing with respect to \( \xi_A \sim \xi_H \). Then referring to Eq. (3), the inverse credibility distribution of \( \xi \) is

\[
\xi = (0.6746, 0.8, 0.8896), \quad \xi = (0.0083, 0.05, 0.0917), \\
\xi = (0.0083, 0.05, 0.0917), \quad \xi = (0.0017, 0.01, 0.0183), \\
\xi = (0.0033, 0.02, 0.0367), \quad \xi = (0.0033, 0.02, 0.0367).
\]

Table 3

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Basic event</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Operator 1 fails to wear safety glasses</td>
</tr>
<tr>
<td>B</td>
<td>Operator 2 fails to wear safety glasses</td>
</tr>
<tr>
<td>C</td>
<td>Machine 1 is operating</td>
</tr>
<tr>
<td>D</td>
<td>Machine 2 is operating</td>
</tr>
<tr>
<td>E</td>
<td>Persons enter the area without safety glasses</td>
</tr>
<tr>
<td>F</td>
<td>Persons enter the area bringing material</td>
</tr>
<tr>
<td>G</td>
<td>Persons enter area carrying away made product</td>
</tr>
<tr>
<td>H</td>
<td>Persons enter the area for other reasons</td>
</tr>
</tbody>
</table>

Fig. 10. The fault tree and possibilities of occurrence of basic events.
obtained as
\[
\Phi_1^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \Phi_3^{-1}(\alpha), \Phi_4^{-1}(\alpha)),
\]
\[
\Phi_2^{-1}(\alpha), \Phi_3^{-1}(\alpha), \Phi_4^{-1}(\alpha), \Phi_5^{-1}(\alpha))
\]
\[
\begin{cases}
1 - (-0.0334\alpha + 0.9967)^2(1 - (1 - \alpha)) \leq \alpha < 0.5 \\
1 - (-0.0334\alpha + 0.9967)^2(1 - (1 - \alpha)) \leq \alpha < 0.5 \\
(1.6666\alpha + 0.1667),
\end{cases}
\]
\[
if \ 0 \leq \alpha < 0.5
\]
\[
if \ 0.5 \leq \alpha \leq 1.
\]
\[
(10)
\]

Afterwards, by using the 'polyfit' function of Matlab and Theorem 3, a relatively exact membership function of \( \xi \) is finally figured out and depicted in Figure 11 as a solid line.

Additionally, for comparison, the other three approaches are utilized to deal with fuzzy arithmetic in this technical example as well. The first one is an approximation method from Singer [26], in which the add and multiplication operation on two TFNs \( \xi_1 = (a_1, b_1, c_1) \) and \( \xi_2 = (a_2, b_2, c_2) \) are defined by
\[
\xi_1 \oplus \xi_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2),
\]
\[
\xi_1 \odot \xi_2 = (a_1 b_2 + a_2 b_1 - b_1 b_2, b_1 b_2, c_1 b_2 + c_2 b_1 - b_1 b_2).
\]

The second one is the standard approximation method employed by Chen [7]. Besides, the interval arithmetic approach is also adopted to calculate the reliability of the grinding machine system. The results of the membership functions calculated by the three methods are presented graphically in Figure 11 as well, in which the results in Singer [26] and Chen [7] are depicted by the dotted line and the dot-dashed line, respectively, and the \( h \)-cuts derived from the interval arithmetic approach are shown with ‘•’.

As shown in Figure 11, all the points indicating the left and right limit values of \( h \)-cuts of \( \xi \) just fall upon the solid line when \( h = 0, 0.05, \cdots, 1 \), which gives a strong proof for the correctness of the membership function we derived through the proposed inverse distribution approach. Although the mean value and the spreads of \( \xi \) deduced by Chen’s [7] method, the interval arithmetic approach and our method are completely equal, the solution accuracy of the standard approximation method employed by Chen [7] is apparently inferior to the other two methods, especially when the system reliability takes value within 0.0187 and 0.1378. As to the approximation method utilized by Singer [26], the solution accuracy performs worse compared with the results obtained by Chen [7].

Further, for demonstrating the performance of the four methods under different parameter settings, we change \( \xi_C \), \( \xi_D \) and \( \xi_E \) as \((0.1333, 0.8, 1), (0.1333, 0.8, 1) \) and \((0.1667, 1, 1) \), respectively. The aforementioned analysis procedure is homoplastically repeated, and then the obtained results are reported in Figure 12 for comparison. Similar conclusions could be deduced as well from Figure 12, in which the solution accuracy by Chen’s [7] method becomes worse with respect to the new parameters. Moreover, Singer’s [26] method even possibly returns a negative reliability in some cases, which distinctly reveals low-accuracy and impropriety of their approximation approach from the angle of real applications.

To sum up, the proposed inverse distribution approach is applied to a system reliability analysis problem related to grinding machine safety in this section. The numerical results demonstrate the efficiency and effectiveness of our method. All comparative study with the other three methods in the current literature is also conducted for different parameter settings, which shows the high accuracy of the proposed novel method.
6. Conclusion

In this paper, we have contributed to the research area of fuzzy arithmetic approach in the following three aspects: (i) for any fuzzy number $\xi$ with a continuous and strictly increasing credibility distribution including TFN as a special case, some theorems on relations among its membership function $\mu_\xi$, credibility distribution $\Phi_\xi$, and inverse credibility distribution $\Phi_\xi^{-1}$ were proved as the theoretical fundamental of our proposed approach. (ii) a novel inverse distribution approach was developed for deriving the membership function $\mu_\xi$ of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ from the membership functions $\mu_{\xi_1}, \mu_{\xi_2}, \ldots, \mu_{\xi_n}$ of $\xi_1, \xi_2, \ldots, \xi_n$, in which the involved fuzzy numbers $\xi_1, \xi_2, \ldots, \xi_n$ are with continuous and strictly increasing credibility distributions, and $f$ is continuous and strictly monotone. The detailed procedures of this approach were described step by step by taking TFNs for instance due to its extensive applications while expressing the ambiguous valuations in practical problems. It has been further pointed out in Section 3.3 that our method is useful when dealing with fuzzy arithmetics in most fuzzy optimization problems in practice since almost all objective and constraint functions in practical problems are indeed monotone with respect to the fuzzy parameters. (iii) Based on the principle whether the inverse function can be easily derived or not, some non-complicated and complicated membership functions with respect to TFNs were given to illustrate the effectiveness of the proposed method in comparison with the standard approximation method and the interval arithmetic approach. An example of the grinding machine system reliability analysis was also presented to show the performance of our method in real applications. All the numerical results demonstrated that a considerable improvement could be achieved by our approach compared with the previous interval arithmetic approach or the standard approximation method, outputting the exact or a well approximate membership function with an extremely high accuracy for the subsequent applications.

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References